# Chapter 9

## Perturbations in the Universe

In this chapter the theory of linear perturbations in the universe are studied.

## 9.1 Differential Equations of Linear Perturbation in the Universe

A covariant, linear, cosmological perturbation theory is given. The metric is the pseudo-Euclidean geometry. The energy-momentum tensor is stated and the basic equations for the propagation of the perturbations are presented. The perturbed equations for a homogeneous isotropic universe are stated. All the results of this chapter can be found in [$Pet\ 95b$].

We use the pseudo-Euclidean geometry (1.5), the theory of gravitation in flat space-time (1.23), the equations of motion (1.29) and the conservation of the whole energy-momentum (1.25). The matter tensor is given by (1.28).

The gravitational field satisfies

$$g^{ij} \to g^{ij} + \Delta g^{ij} \tag{9.1a}$$

with the condition

$$\left|\Delta g^{ij}\right| \ll \left|g^{ij}\right|. \tag{9.1b}$$

It follows by linear perturbation

$$g_{ij} \to g_{ij} + \Delta g_{ij} \tag{9.2a}$$

with the result

$$\Delta g_{ij} = -g_{ik} g_{jl} \Delta g^{kl}. \tag{9.2b}$$

In addition, we put

$$\rho \to \rho + \Delta\rho, p \to p + \Delta p, u^i \to u^i + \Delta u^i. \tag{9.3}$$

The arising equations of perturbations are applied to cosmological models with

$$\begin{aligned} g^{ij} &= 1/a^2(t) \ \ (i = j = 1,2,3) \\ &= -h(t) \ \ (i = j = 4) \\ &= 0 \ \ (i \neq j) \end{aligned}$$

as considered in chapter VII.

Let $(\Delta v^1, \Delta v^2, \Delta v^3)$ be the perturbed velocity. Put the perturbed potentials

$$f = a^2 \Delta g^{ii}, d = -\frac{1}{h}\Delta g^{44}, b_i = a^2 \Delta g^{i4} \quad \text{(i=1,2,3)}. \tag{9.4}$$

Then, the cosmological model implies after longer calculations the differential equations for the perturbed field

(1) $$\sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{\partial f}{\partial x^k}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial f}{\partial t}\right) = -\frac{2}{c}\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{a'}{a}b_k\right)$$
$$-\frac{3}{c^2}a^3\sqrt{h}\frac{a'}{a}\left(\frac{\partial f}{\partial t} + \frac{\partial d}{\partial t}\right) - 3\kappa c^2(\rho - p)(f + d) + 2\kappa c^2(\Delta\rho - \Delta p)$$

(2) $$\sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{\partial b_i}{\partial x^k}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial b_i}{\partial t}\right) = -\frac{1}{c^2}a^3\sqrt{h}\frac{\partial b_i}{\partial t}\left(2\frac{a'}{a} + \frac{h'}{h}\right)$$
$$-\frac{1}{c^2}a^3\sqrt{h}b_i\left(-3\left(\frac{a'}{a}\right)^2 + 3\frac{a'}{a}\frac{h'}{h} + \frac{1}{4}\left(\frac{h'}{h}\right)^2\right)$$
$$+\frac{3}{2}\frac{1}{c}a^3\sqrt{h}\left(-\frac{a'}{a} + \frac{1}{2}\frac{h'}{h}\right)\frac{\partial f}{\partial x^i} - \frac{1}{2}\frac{1}{c}a^3\sqrt{h}\left(3\frac{a'}{a} + \frac{1}{2}\frac{h'}{h}\right)\frac{\partial d}{\partial x^i} +$$
$$4\kappa c^2(\rho + p)a^2 h\frac{\Delta v^i}{c} \quad \text{(i=1,2,3)}$$

$$\sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{\partial d}{\partial x^k}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial d}{\partial t}\right)$$
(3) $$= -\frac{1}{c}\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{h'}{h}b_k\right)$$
$$+\frac{3}{c^2}a^3\sqrt{h}\left(-\frac{a'}{a} + \frac{h'}{h}\right)\frac{\partial f}{\partial t} + \frac{1}{c^2}a^3\sqrt{h}\left(-3\frac{a'}{a} + \frac{h'}{h}\right)\frac{\partial d}{\partial t}$$
$$+3\kappa c^2(\rho + 3p)(f + d) - 2\kappa c^2(\Delta\rho + 3\Delta p). \tag{9.5}$$

The perturbed equations of motion are:

(1) $$\frac{\partial \Delta p}{\partial x^i} + \frac{1}{c}\frac{\partial}{\partial t}\left\{(\rho + p)\left(b_i + a^2 h\frac{\Delta v^i}{c}\right)\right\} = -\frac{3}{2}p\frac{\partial f}{\partial x^i} + \frac{1}{2}p\frac{\partial d}{\partial x^i} \quad \text{(i=1,2,3)}$$

(2) $$-(\rho + p)\sum_{k=1}^{3}\frac{\partial \Delta v^k}{\partial x^k} - \frac{\partial \Delta p}{\partial t} = -\frac{3}{2}p\frac{\partial f}{\partial t} + \frac{1}{2}\rho\frac{\partial d}{\partial t} + 3\frac{a'}{a}\Delta p + \frac{1}{2}\frac{h'}{h}\Delta\rho. \tag{9.6}$$

Furthermore, we have an equation of state for the perturbed pressure. i.e.,

$$\Delta p = p(\Delta\rho). \tag{9.7}$$

The relations (9.5), (9.6) and (9.7) are ten equations for the ten unknown functions $f, b_i$ (i=1,2,3), $d, \Delta v^i$ (i=1,2,3), $\Delta p$ and $\Delta\rho$. These equations describe small perturbations in a homogeneous, isotropic cosmological model.

In the following let us assume that the equation of state has the form

$$\Delta p = v_s^2 \Delta\rho \tag{9.8}$$

with constant velocity sound $0 \leq v_s \leq 1$. It follows as consequence of the perturbed field equation (9.5) and the perturbed equations of motion (9.6) a conservation law of the perturbed energy-momentum tensor (see $[Pet\ 95b]$).

## 9.2 Spherically Symmetric Perturbations

We will now study spherically symmetric solutions of the perturbed equations (9.5), (9.6) and (9.8). The study of these results are contained in the following sub-chapters and are found in the articles $[Pet\ 96a]$ and $[Pet\ 96b]$.

Let $r$ denote the Euclidean distance from the centre of the spherical symmetry and let $k$ be the wave number. We make the ansatz

$$f(r,t) \to \tilde{f}(k,t)\frac{sin(kr)}{r},\ d(r,t) \to \tilde{d}(k,t)\frac{sin(kr)}{r},$$

$$b_i(r,t) \to \frac{c}{H_0}\tilde{b}(k,t)\frac{\partial}{\partial x^i}\left(\frac{sin(kr)}{r}\right), \tag{9.9}$$

$$\Delta\rho(r,t) \to \tilde{\rho}(k,r)\frac{sin(kr)}{r}, \Delta v^i(r,t) \to \frac{c}{H_0}\tilde{v}(k,t)\frac{\partial}{\partial x^i}\left(\frac{sin(kr)}{r}\right).$$

We get by substituting the relations (9.9) into the equations (9.5) and (9.6)

(1) $$-k^2\frac{a}{\sqrt{h}}\tilde{f} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial\tilde{f}}{\partial t}\right) = -\frac{2}{c}k^2\frac{a}{\sqrt{h}}\frac{a'}{a}\frac{c}{H_0}\tilde{b} - \frac{3}{c^2}a^3\sqrt{h}\frac{a'}{a}\left(\frac{\partial f}{\partial t} + \frac{\partial\tilde{d}}{\partial t}\right)$$

$$-3\kappa c^2(\rho - p)(\tilde{f} + \tilde{d}) + 2\kappa c^2(1 - v_s^2)\tilde{\rho},$$

(2) $$-k^2\frac{a}{\sqrt{h}}\tilde{b} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial\tilde{b}}{\partial t}\right) = -\frac{1}{c^2}a^3\sqrt{h}\frac{\partial\tilde{b}}{\partial t}\left(2\frac{a'}{a} + \frac{h'}{h}\right)$$

$$-\frac{1}{c^2}a^3\sqrt{h}\tilde{b}\left(-3\left(\frac{a'}{a}\right)^2 + 3\frac{a'}{a}\frac{h'}{h} + \frac{1}{4}\left(\frac{h'}{h}\right)^2\right)$$

$$+\frac{3}{2}\frac{H_0}{c^2}a^3\sqrt{h}\left(-\frac{a'}{a}+\frac{1}{2}\frac{h'}{h}\right)\tilde{f}-\frac{1}{2}\frac{H_0}{c^2}a^3\sqrt{h}\left(3\frac{a'}{a}+\frac{1}{2}\frac{h'}{h}\right)\tilde{d}$$

$$+4\kappa c^2(\rho+p)a^2\sqrt{h}\frac{\tilde{v}}{c},$$

(3)

$$-k^2\frac{a}{\sqrt{h}}\tilde{d}-\frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial\tilde{d}}{\partial t}\right)=\frac{1}{c}k^2\frac{a}{\sqrt{h}}\frac{h'}{h}\frac{c}{H_0}\tilde{b}$$

$$+\frac{3}{c^2}a^3\sqrt{h}\left(-\frac{a'}{a}+\frac{h'}{h}\right)\frac{\partial\tilde{f}}{\partial t}+\frac{1}{c^2}a^3\sqrt{h}\left(-3\frac{a'}{a}+\frac{h'}{h}\right)\frac{\partial\tilde{d}}{\partial t} \tag{9.10}$$

$$+3\kappa c^2(\rho+3p)\left(\tilde{f}+\tilde{d}\right)-2\kappa c^2(1+3v_s^2)\tilde{\rho}.$$

(4) $$v_s^2\tilde{\rho}+\frac{1}{c}\frac{\partial}{\partial t}\left\{(\rho+p)\frac{c}{H_0}\left(\tilde{b}+a^2h\frac{\tilde{v}}{c}\right)\right\}=-\frac{3}{2}p\tilde{f}+\frac{1}{2}\rho\tilde{d},$$

(5) $$(\rho+p)k^2\frac{c^2}{H_0}\frac{\tilde{v}}{c}-\frac{\partial\tilde{\rho}}{\partial t}=-\frac{3}{2}p\frac{\partial\tilde{f}}{\partial t}+\frac{1}{2}\rho\frac{\partial\tilde{d}}{\partial t}+\left(3v_s^2\frac{a'}{a}+\frac{1}{2}\frac{h'}{h}\right)\tilde{\rho}. \tag{9.11}$$

The three perturbed field equations (9.10) and the two perturbed equations of motion (9.11) are five linear homogeneous differential equations for the five unknown functions $\tilde{f},\tilde{b},\tilde{d},\tilde{\rho},\tilde{v}$ depending on $t$ and on a parameter $k$. Knowing a solution of (9.10) and (9.11) for $k$ on a fixed interval $I$ we can get a more general solution by virtue of the linearity of the equations. Let $B(k)$ be a function of $k$ on the interval $I$ and let $r_0$ be a fixed distance from the centre then we get the more general solutions

$$\begin{aligned}
f(r,t)&=\int B(k)\frac{\sin(kr)}{r}r_0^2\tilde{f}(k,t)dk,\\
b_i(r,t)&=\int B(k)\frac{\partial}{\partial x^i}\left(\frac{\sin(kr)}{r}\right)r_0^2\frac{c}{H_0}\tilde{b}(k,t)dk,\\
d(r,t)&=\int B(k)\frac{\sin(kr)}{r}r_0^2\tilde{d}(k,t)dk,\\
\Delta\rho(r,t)&=\int B(k)\frac{\sin(kr)}{r}r_0\tilde{\rho}(k,t)dk,\\
\Delta v^i(r,t)&=\int B(k)\frac{\partial}{\partial x^i}\left(\frac{\sin(kr)}{r}\right)r_0\frac{c}{H_0}\tilde{v}(k,t)dk.
\end{aligned} \tag{9.12}$$

Here, the integration is taken over the interval $I$.

In the following we will only consider cosmological models with $\Lambda=0$, i.e.

$$\Omega_m\approx 1 \tag{9.13}$$

and the case

$$v_s^2 = 0. \tag{9.14}$$

We put

$$\varepsilon = \frac{\rho_{r0}}{\rho_{m0}}. \tag{9.15}$$

## 9.3 Beginning of the Universe

The beginning of the universe in flat space-time theory of gravitation is non-singular. All the energy is in form of gravitational energy and radiation and dust arise out of gravitational energy whereas the whole energy is conserved. Put for $t \to -\infty$

$$\xi = +\frac{3}{2}H_0 t - 1. \tag{9.16}$$

Then, we have by (7.31) and (7.14b)

$$\begin{aligned} &a(-\infty) \approx 1.81 a_1, \sqrt{h(t)} \approx \xi^2 / a^3(-\infty), \\ &\rho(t) \approx \rho_{r0} a^2(-\infty) / \xi^2, \; p(t) \approx \frac{1}{3} \rho_{r0} a^2(-\infty) \Big/ \xi^2. \end{aligned} \tag{9.17}$$

In the beginning of the universe the density of matter is negligible and only the density of radiation and its pressure dominate.

We make the ansatz

$$\begin{aligned} \tilde{f} &= \sum_{k=0}^{\infty} \frac{f_{2k}}{\xi^{2k+l}}, \tilde{d} = \sum_{k=0}^{\infty} \frac{d_{2k}}{\xi^{2k+l}}, \tilde{\rho}/\rho_{m0} = \sum_{k=0}^{\infty} \frac{\rho_{2k}}{\xi^{2k+l+2}}, \\ \tilde{b} &= \sum_{k=0}^{\infty} \frac{b_{2k}}{\xi^{2k+l-1}}, \; \frac{\tilde{v}}{c} = a^4(-\infty) \sum_{k=0}^{\infty} \frac{v_{2k}}{\xi^{2k+l+3}}. \end{aligned} \tag{9.18}$$

We get by the substitution of the relations (9.16), (9.17) and (9.18) into the equations (9.10) and (9.11) and by the use of (9.13), (9.14) and (9.15) five homogeneous linear equations to determine $l$ such that not all of the five coefficients $f_0, b_0, d_0, \rho_0, v_0$ vanish. There exist four non-negative values of $l$:

$$l = 0, 1, 2, 5. \tag{9.19}$$

In the following only the case $l = 0$ is studied implying two arbitrary constants $f_0$ and $b_2$. We get

$$b_0 = 0, d_0 = 3f_0, v_0 = \frac{1}{2} f_0,$$
$$\rho_0 = 3 f_0 a^2(-\infty)\left( \varepsilon - \frac{3}{10}\left( \frac{2ck}{3H_0} a(-\infty) \right)^2 \right). \quad (9.20a)$$

Furthermore, it follows

$$f_2 = f_0 a^2(-\infty)\left( \frac{2}{3}\varepsilon + \frac{1}{10}\left( \frac{2ck}{3H_0} a(-\infty) \right)^2 \right),$$
$$d_2 = f_0 a^2(-\infty)\left( -\frac{2}{3}\varepsilon + \frac{3}{10}\left( \frac{2ck}{3H_0} a(-\infty) \right)^2 \right),$$
$$v_2 = -b_2 + \frac{1}{12} f_0 a^2(-\infty)\left( -\frac{4}{3}\varepsilon + \frac{1}{5}\left( \frac{2ck}{3H_0} a(-\infty) \right)^2 \right), \quad (9.20b)$$
$$\rho_2 = \frac{2}{3}\varepsilon f_0 a^4(-\infty)\left( \varepsilon + \frac{3}{5}\left( \frac{2ck}{3H_0} a(-\infty) \right)^2 \right).$$

All the other coefficients can be recursively calculated. Hence, we get for $l = 0$ the solution (9.18) depending on two arbitrary parameters. Then, the relations (9.12) give the perturbed solutions in the beginning of the universe. Let us now discuss the received perturbed solution. Assuming $B(k) \geq 0$ on the interval $I$ we have to put $f_0 > 0$ to get collapsing spherically symmetric perturbations in the neighbourhood of the centre as $t$ increases from $-\infty$. This result follows by the use of (9.20a), (9.18) and (9.12). Furthermore, we get that the density of the spherically symmetric perturbation is positive if the wave numbers fulfil the condition

$$0 \leq k \leq \left(\frac{10}{3}\varepsilon\right)^{1/2} \frac{3H_0}{2c} \frac{1}{a(-\infty)}. \quad (9.21)$$

Hence, we have to the lowest order of the density fluctuations as $t \to -\infty$

$$\Delta\rho(r,t)/\rho_m(t) \approx \int B(k) \frac{\sin(kr)}{r} r_0^2 3 \frac{f_0}{a(-\infty)} \left( \varepsilon - \frac{3}{10}\left( \frac{2ck}{3H_0} a(-\infty) \right)^2 \right) dk. \quad (9.22)$$

Therefore, for the case $l = 0$ small spherically symmetric non-homogeneities in the uniform distribution of matter can exist in the beginning of the universe.

The cases $l = 1, 2, 5$ give only one-parametric solutions with

$$\Delta\rho(r,t)/\rho_m(t) = O\left(\frac{1}{\xi^2}\right)$$

for $t \to -\infty$.

Therefore, small non-homogeneities can arise in the homogeneous distribution of matter in the beginning. By virtue of the small horizons there are many unconnected regions in the universe. The non-homogeneities are unconnected and arise independently from one another. Therefore, they are uniformly distributed in space in the beginning of the universe. This may explain the presently observed homogeneity of matter on large scales in the universe. The horizons increase in the course of time and larger regions of the universe become connected. The non-homogeneities are then connected and influence one another by gravitation.

## 9.4 Matter Dominated Universe

In this sub-chapter the universe is considered where matter dominates radiation. Put

$$\xi = \left(\frac{3}{2}H_0(t - t_1)\right)^{1/3} \tag{9.23}$$

with

$$H_0 t_1 \approx -\frac{2}{3}. \tag{9.24}$$

During the studied time epoch it holds by (7.46) and (7.14b)

$$h(t) \approx 1, \ \ a(t) \approx \xi^2, \ \ \rho(t) \approx \rho_{m0}, \ \ \ p \approx 0. \tag{9.25}$$

We make the ansatz

$$\tilde{f} = \xi^l \textstyle\sum_{k=0}^{\infty} f_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}, \ \ \tilde{d} = \xi^l \textstyle\sum_{k=0}^{\infty} d_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k},$$

$$\tilde{b} = \xi^{l+3} \textstyle\sum_{k=0}^{\infty} b_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}, \ \ \frac{\tilde{\rho}}{\rho_{m0}} = \xi^l \int_{k=0}^{\infty} \rho_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}, \tag{9.26}$$

$$\frac{\tilde{v}}{c} = \xi^{l-1} \textstyle\sum_{k=0}^{\infty} v_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}.$$

It follows in analogy to the previous sub-chapter

$$l = 0, 9 \tag{9.27}$$

which imply non-vanishing solutions. Furthermore, a pair of complex numbers is received to get non-vanishing solutions. The case

$$l = 9 \tag{9.28}$$

is further studied. It follows with an arbitrary parameter $\rho_0$:

$$f_0 = -\frac{13}{3}\rho_0,\ d_0 = -2\rho_0,\ b_0 = -\frac{17}{60}\rho_0,\ v_0 = \frac{7}{60}\rho_0. \tag{9.29a}$$

The coefficients of higher order can again be recursively calculated. It can be proved that the series (9.26) converge absolutely and uniformly. Hence, the sums and the integrals of (9.12) can be exchanged. Put $I = [0, k_0]$ with $k_0$ sufficiently small, i.e. large scale non-homogeneities we have to the lowest order

$$\Delta\rho(r,t)/\rho_m(t) \approx \rho_0 r_0^2 a(t)^{9/2} \int B(k)\frac{sin(kr)}{r}dk. \tag{9.29b}$$

This solution is non-singular for $r = 0$ whereas in $[Isr\ 94]$ spherically symmetric perturbations are considered by the use of general relativity yielding a singularity at $r = 0$. Hence, the density contrast in the matter dominated universe increases faster than by the use of general relativity (see e.g. $[Bar\ 80]$, $[Isr\ 94]$). In these articles it is proved that the density contrast increases at most linearly with the function $a(t)$.

Let $t_d$ be the time of the decoupling of matter and radiation. Then, relation (9.29b) yields

$$\Delta\rho(r,t)/\rho_m(t) \approx \Delta\rho(r,t_d)/\rho_m(t_d)\left(\frac{a(t)}{a(t_d)}\right)^{9/2}. \tag{9.30}$$

It holds for adiabatic perturbations

$$|\Delta\rho(r,t_d)/\rho_m(t_d)| \approx 3|\Delta T/T|_d. \tag{9.31}$$

Here, $\Delta T$ denotes the temperature anisotropy of CMBR. The decoupling occurs at a redshift $z_d$ (see e.g.,$[Kol\ 90]$)

$$1/a(t_d) = 1 + z_d \approx 1100. \tag{9.32}$$

The analysis of COBE-data show that the CMBR has an anisotropy of

$$|\Delta T/T|_d \approx 10^{-5} \tag{9.33}$$

on large scales [$Smo$ 92]. Hence, relation (9.30) gives by the use of (9.31), (9.32) and (9.33)

$$|\Delta\rho(r,t)/\rho_m(t)| \approx 3 \cdot 10^{-5}\big(1100 \cdot a(t)\big)^{9/2}. \tag{9.34}$$

The time $\bar{t}$ where the density contrast is given by

$$|\Delta\rho(r,\bar{t})/\rho_m(\bar{t})| \approx 1$$

implies a redshift $\bar{z}$ with

$$1 + \bar{z} = 1/a(\bar{t}) \approx 1100 \cdot \left(\frac{3}{10^5}\right)^{2/9} \approx 108. \tag{9.35}$$

Summarizing, large scale structures can arise in the matter dominated universe in accordance with the observed CMBR anisotropy. It is worth to mention that for a density contrast greater than one non-linear perturbations must be considered.

All these results with detailed calculations are given in the articles of [$Pet$ 95$a$] and [$Pet$ 96$a$, 96$b$] where also further remarks can be found.

Spherically symmetric perturbations in a universe which contains an additional field as source are studied in the article [$Pet$ 97$a$].

In the paper [$Gro$ 97] higher order approximations of density perturbations are given as well in the beginning as in the matter dominated universe. The results are based on numerical computations. Numerical computations of spherically symmetric density perturbations in a universe with an additional field are stated in the paper [$Sch$ 97].

For the study of the early universe and structure formation by the use of Einstein's theory, e.g., the books of [$Kol$ 90] [$Pee$ 80] and [$Pad$ 93] shall be considered.

It should also be remarked that the theory of Einstein implies a too small density contrast which yields difficulties to explain the large scale structures in the universe as galaxies, etc.