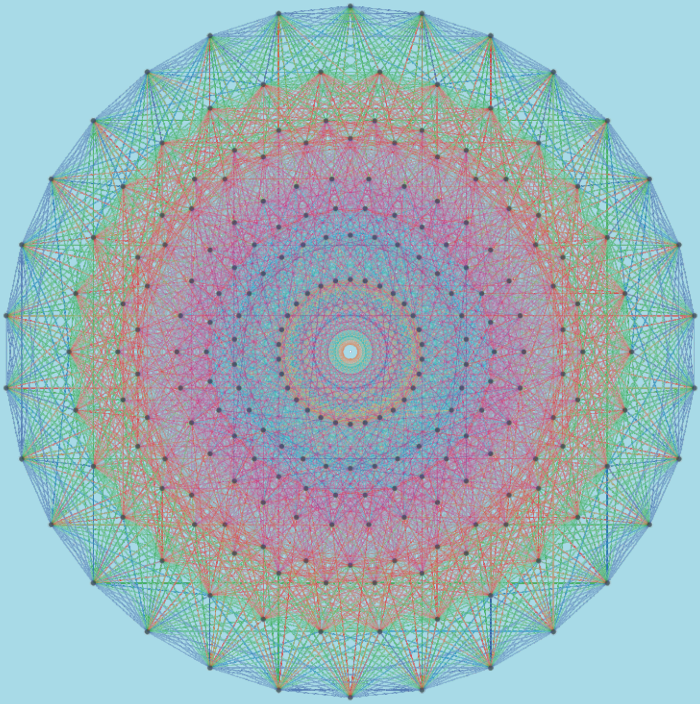

Axiomatic K-theory for C*-algebras

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Preface

In *Part I* we present an axiomatic frame in which many results of the K-theory for C*-algebras are proved. In *Part II* we construct an example for this axiomatic theory, which generalizes the classical theory for C*-algebras. This last theory starts by associating to each C*-algebra F the C*-algebras of square matrices with entries in F. Every such C*-algebra of square matrices can be obtained as the projective representation of a certain group with respect to a Schur function for this group with values in $\mathbb{C}$ (Definition 5.0.1). The above mentioned generalization consists in replacing this Schur function by an arbitrary Schur function which satisfies some axiomatic conditions. Moreover this Schur function can take its values in a commutative unital C*-algebra E instead of $\mathbb{C}$. In this case this K-theory does not apply to the category of C*-algebras, but to the category of E-C*-algebras (Definition 1.1.1), which are C*-algebras endowed with a supplementary structure (every C*-algebra can be endowed with such a supplementary structure (Proposition 1.1.3)). Up to some definitions and notation Part II is independent of *Part I*.

In general we use the notation and the terminology of [1]. In the sequel we give a list of notation used in this book.

1) $\mathbb{C}$ (respectively $\mathbb{R}$) denotes the field of complex (respectively real) numbers, $\mathbb{N}$ denotes the set of natural numbers ($0 \notin \mathbb{N}$), $\mathbb{N}^* := \mathbb{N} \cup \{0\}$, $\mathbf{Z}$ denotes the group of integers, and for every $n \in \mathbb{N}^*$ we put $\mathbb{N}_n := \{\, k \in \mathbb{N} \mid k \leq n \,\}$ and $\mathbf{Z}_n := \mathbf{Z}/(n\mathbf{Z})$.
2) For every set A, *Card* A denotes the cardinal number of A and id_A denotes the identity map of A. If x is a map defined on A and B is a subset of A then $x|B$ denotes the restriction of x to B.
3) Let $(\Omega_j)_{j\in J}$ be a family of topological spaces and let Ω be the disjoint union of this family. The topological sum of the family $(\Omega_j)_{j\in J}$ is the topological space obtained by endowing Ω with the topology $\left\{\, U \subset \Omega \mid j \in J \Rightarrow U \cap \Omega_j \text{ is an open set of } \Omega_j \,\right\}$.
4) If Ω is a topological space and G is a C*-algebra then $\mathscr{C}(\Omega, G)$ denotes the C*-algebra of continuous bounded maps of Ω into G (endowed with the supremum norm). If Ω is a locally compact space then $\mathscr{C}_0(\Omega, G)$ denotes the C*-algebra of continuous maps of Ω into G vanishing at the infinity.
5) $\odot$ denotes the algebraic tensor product of vector spaces.
6) $\approx$ means isomorphic.

Contents

Throughout this book E denotes a fixed commutative unital C*-algebra.

Part I

Axiomatic K-theory

Throughout *Part I* we endow $\{0,1\}$ with the structure of a group by identifying it with $\mathbf{Z}_2$ and take $i \in \{0,1\}$.

Chapter 1

The Axiomatic Theory

1.1 E-C*-algebras

DEFINITION 1.1.1 *In this book we call E***-C*-algebra** *a C*-algebra F endowed with a bilinear map (exterior multiplication)*

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

such that for all $\alpha, \beta \in E$ and $x, y \in F$,

$$(\alpha+\beta)x = \alpha x + \beta y, \quad (\alpha\beta)x = \alpha(\beta x), \quad (\alpha x)^* = \alpha^* x^*, \quad \|\alpha x\| \leq \|\alpha\| \, \|x\|,$$

$$\alpha(x+y) = \alpha x + \alpha y, \qquad \alpha(xy) = (\alpha x)y = x(\alpha y), \qquad 1_E x = x.$$

*An E***-C*-subalgebra (***E***-ideal)** *of F is a C*-subalgebra (a closed ideal) G of F such that*

$$(\alpha, x) \in E \times G \Longrightarrow \alpha x \in G.$$

If F, G are E-C-algebras then a C*-homomorphism $\varphi : F \longrightarrow G$ is called E***-linear** *or an E***-C*-homomorphism** *if for all $(\alpha, x) \in E \times F$, $\varphi(\alpha x) = \alpha\varphi x$. A bijective E-C*-homomorphism is called E***-C*-isomorphism***. We denote by 0 the E-C*-algebra having a unique element. We denote by $\mathfrak{M}_E$ the category of E-C*-algebras for which the morphisms are the E-linear C*-homomorphisms. In particular $\mathfrak{M}_{\mathbf{C}}$ is the category of all C*-algebras.*

If G is an E-ideal of the E-C-algebra F then the C*-algebra F/G has a natural structure of an E-C*-algebra and*

$$0 \longrightarrow G \xrightarrow{\varphi} F \xrightarrow{\psi} F/G \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$, where φ denotes the inclusion map and ψ the quotient map. Conversely, if

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ then F is an E-ideal of G and $H \approx G/F$.

DEFINITION 1.1.2 *If $(F_j)_{j\in J}$ is a finite family of E-C*-algebras then we denote by $\prod\limits_{j\in J} F_j$ the E-C*-algebra obtained by endowing the corresponding C*-algebra $\prod\limits_{j\in J} F_j$ with the bilinear map*

$$E \times \prod_{j\in J} F_j \longrightarrow \prod_{j\in J} F_j, \quad (\alpha, (x_j)_{j\in J}) \longmapsto (\alpha x_j)_{j\in J}.$$

PROPOSITION 1.1.3 *Every C*-algebra can be endowed with the structure of an E-C*-algebra.*

Let F be a C*-algebra. Let Ω be the spectrum of E and $\omega \in \Omega$ and put

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha(\omega)x\,.$$

It is easy to see that F endowed with this exterior multiplication is an E-C*-algebra. ■

EXAMPLE 1.1.4 *Let Ω be a finite set and $E := \mathscr{C}(\Omega, \mathbb{C})$.*

a) Let $(F_\omega)_{\omega\in\Omega}$ be a finite family of C-algebras and $F := \prod\limits_{\omega\in\Omega} F_\omega$. If we put for all $(\alpha, x) \in E \times F$,*

$$\alpha x : \Omega \longrightarrow F\,, \quad \omega \longmapsto \alpha(\omega)x_\omega$$

then F endowed with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

is an E-C-algebra.*

b) Let F be an E-C-algebra and for every $\omega \in \Omega$ put*

$$e_\omega : \Omega \longrightarrow \mathbb{C}\,, \quad \omega' \longmapsto \begin{cases} 1 & \text{if} \quad \omega' = \omega \\ 0 & \text{if} \quad \omega' \neq \omega \end{cases}\,,$$

$$F_\omega := \{\, e_\omega x \mid x \in F \,\}\,.$$

Then F_ω is a C-algebra for all $\omega \in \Omega$ and $F \approx \prod\limits_{\omega\in\Omega} F_\omega$, with the meaning of a).* ■

EXAMPLE 1.1.5 *Let Ω be a discrete locally compact space, Ω^* a compactification of Ω, $E := \mathscr{C}(\Omega^*, \mathbb{C})$, $(F_\omega)_{\omega\in\Omega}$ a family of C*-algebras, and $F := \prod\limits_{\omega\in\Omega} F_\omega$ $\left(\text{resp. } F := \left\{ x \in \prod\limits_{\omega\in\Omega} F_\omega \;\middle|\; \lim_{\omega\to\infty} \|x_\omega\| = 0 \right\}\right)$. If we put for all $(\alpha, x) \in E \times F$*

$$\alpha x : \Omega \longrightarrow F\,, \quad \omega \longmapsto \alpha(\omega)x_\omega$$

then $\alpha x \in F$ for all $(\alpha, x) \in E \times F$ and F endowed with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

is an E-C-algebra.* ■

1.2 The Axioms

DEFINITION 1.2.1 *We denote by K_0 and K_1 two covariant functors from the category $\mathfrak{M}_E$ to the category of additive groups. We denote by 0 the group which has a unique element and call* **K-null** *an E-C*-algebra F for which $K_i(F) = 0$. Let $F \overset{\varphi}{\to} G$ be a morphism in $\mathfrak{M}_E$. We say that φ is* **K-null** *if $K_i(\varphi) = 0$. We say that φ* **factorizes through null** *if there are morphisms $F \overset{\varphi'}{\to} H$ and $H \overset{\varphi''}{\to} G$ in $\mathfrak{M}_E$ such that $\varphi = \varphi'' \circ \varphi'$ and such that H is K-null.*

We have $K_i(id_F) = id_{K_i(F)}$ for every E-C*-algebra F. Every morphism which factorizes through null is K-null.

AXIOM 1.2.2 (Null-axiom) *$K_i(0) = 0$.*

AXIOM 1.2.3 (Split exact axiom) *If*

$$0 \longrightarrow F \overset{\varphi}{\longrightarrow} G \underset{\overset{\longleftarrow}{\lambda}}{\overset{\psi}{\longrightarrow}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ then

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\varphi)} K_i(G) \underset{\overset{\longleftarrow}{K_i(\lambda)}}{\overset{K_i(\psi)}{\longrightarrow}} K_i(H) \longrightarrow 0$$

is a split exact sequence in the category of additive groups.

It follows that the map

$$K_i(F) \times K_i(H) \longrightarrow K_i(G), \quad (a,b) \longmapsto K_i(\varphi)a + K_i(\lambda)b$$

is a group isomorphism.

DEFINITION 1.2.4 *Let $\varphi, \psi : F \longrightarrow G$ be morphisms in $\mathfrak{M}_E$. We say that φ and ψ are* **homotopic** *if there is a path*

$$\phi_s : F \longrightarrow G, \qquad s \in [0,1]$$

of morphisms in $\mathfrak{M}_E$ such that $\phi_0 = \varphi$, $\phi_1 = \psi$, and the map

$$[0,1] \longrightarrow G, \quad s \longmapsto \phi_s x$$

is continuous for every $x \in F$.

We say that a pair $F \xrightarrow{\varphi} G$, $G \xrightarrow{\psi} F$ of morphisms in $\mathfrak{M}_E$ is a **homotopy** *if $\psi \circ \varphi$ is homotopic to id_F and $\varphi \circ \psi$ is homotopic to id_G. In this case we say that F and G are* **homotopic**. *F is called* **null-homotopic** *if it is homotopic to the E-C*-algebra* 0.

AXIOM 1.2.5 (Homotopy axiom) *If $\varphi, \psi : F \longrightarrow G$ are homotopic morphisms in $\mathfrak{M}_E$ then $K_i(\varphi) = K_i(\psi)$.*

DEFINITION 1.2.6 *We associate to every exact sequence*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

in $\mathfrak{M}_E$ two group homomorphisms (called **index maps)**

$$\delta_i : K_i(H) \longrightarrow K_{i+1}(F) \ .$$

AXIOM 1.2.7 (Six-term axiom) *For every exact sequence in $\mathfrak{M}_E$*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

the six-term sequence

$$\begin{tikzcd}
K_0(F) \arrow[r, "K_0(\varphi)"] & K_0(G) \arrow[r, "K_0(\psi)"] & K_0(H) \arrow[d, "\delta_0"] \\
K_1(H) \arrow[u, "\delta_1"] & K_1(G) \arrow[l, "K_1(\psi)"] & K_1(F) \arrow[l, "K_1(\varphi)"]
\end{tikzcd}$$

is exact.

AXIOM 1.2.8 (Commutativity of the index maps) *If the diagram in $\mathfrak{M}_E$*

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\varphi"] \arrow[d, "\phi_1"'] & G \arrow[r, "\psi"] \arrow[d, "\phi_2"'] & H \arrow[r] \arrow[d, "\phi_3"] & 0 \\
0 \arrow[r] & F' \arrow[r, "\varphi'"'] & G' \arrow[r, "\psi'"'] & H' \arrow[r] & 0
\end{tikzcd}$$

is commutative and has exact rows then the diagram

$$\begin{array}{ccc} K_i(H) & \xrightarrow{\delta_i} & K_{i+1}(F) \\ {\scriptstyle K_i(\phi_3)}\downarrow & & \downarrow{\scriptstyle K_{i+1}(\phi_1)} \\ K_i(H') & \xrightarrow[\delta'_1]{} & K_{i+1}(F') \end{array}$$

is commutative, where δ_i *and* δ'_i *denote the index maps associated to the upper and the lower row of the above diagram, respectively.*

Remark. The above axioms are fulfilled if $K_i(F) = 0$ for all E-C*-algebras F .

1.3 Some Elementary Results

PROPOSITION 1.3.1 *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\lambda}{\overset{\psi}{\rightleftarrows}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ *then its index maps are* 0.

By the split exact axiom (Axiom 1.2.3),

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\varphi)} K_i(G) \underset{K_i(\lambda)}{\overset{K_i(\psi)}{\rightleftarrows}} K_i(H) \longrightarrow 0$$

is a split exact sequence in the category of additive groups and the assertion follows from the six-term axiom (Axiom 1.2.7). ■

DEFINITION 1.3.2 *Let* $(F_j)_{j\in J}$ *be a finite family of* E*-C*-algebras,* $F := \prod_{j\in J} F_j$ *and for every* $j \in J$ *let* $\varphi_j : F_j \longrightarrow F$ *be the canonical inclusion and* $\psi_j : F \longrightarrow F_j$ *the canonical projection. We define*

$$\Phi_{(F_j)_{j\in J},i} : \prod_{j\in J} K_i(F_j) \longrightarrow K_i(F)\,, \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} K_i(\varphi_j)\, a_j\,,$$

$$\Psi_{(F_j)_{j\in J},i} : K_i(F) \longrightarrow \prod_{j\in J} F_j\,, \quad a \longmapsto (K_i(\psi_j)\, a)_{j\in J}\,.$$

PROPOSITION 1.3.3 *If $(F_j)_{j\in J}$ is a finite family of E-C*-algebras then the map*

$$\Phi_{(F_j)_{j\in J},i}:\prod_{j\in J}K_i(F_j)\longrightarrow K_i\left(\prod_{j\in J}F_j\right)$$

is a group isomorphism and

$$\Psi_{(F_j)_{j\in J},i}:K_i\left(\prod_{j\in J}F_j\right)\longrightarrow\prod_{j\in J}K_i(F_j)$$

is its inverse.

If $J=\emptyset$ then the assertion follows from the null-axiom (Axiom 1.2.2). The assertion is trivial for $Card\,J=1$. We prove the general case by induction with respect to $Card\,J$. Let $j_0\in J$ and assume the assertion holds for $J':=J\setminus\{j_0\}$. We denote by

$$\varphi:F_{j_0}\longrightarrow\prod_{j\in J}F_j\,,\qquad\qquad \lambda:\prod_{j\in J'}F_j\longrightarrow\prod_{j\in J}F_j$$

the canonical inclusion maps and by

$$\psi:\prod_{j\in J}F_j\longrightarrow\prod_{j\in J'}F_j$$

the canonical projection. Then

$$0\longrightarrow F_{j_0}\xrightarrow{\varphi}\prod_{j\in J}F_j\underset{\xleftarrow[\lambda]{}}{\xrightarrow{\psi}}\prod_{j\in J'}F_j\longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$. By the split exact axiom (Axiom 1.2.3) the map

$$\Psi_i:K_i\left(F_{j_0}\right)\times K_i\left(\prod_{j\in J'}F_j\right)\longrightarrow K_i\left(\prod_{j\in J}F_j\right),\quad (a,b)\longmapsto K_i(\varphi)\,a+K_i(\lambda)\,b$$

is a group isomorphism. Since

$$\Psi_i\circ\left(id_{K_i\left(F_{j_0}\right)}\times\Phi_{(F_j)_{j\in J'},i}\right)=\Phi_{(F_j)_{j\in J},i}$$

it follows from the induction hypothesis that $\Phi_{(F_j)_{j\in J},i}$ is a group isomorphism.

The last assertion follows from $\psi_j\circ\varphi_j=id_{F_j}$ for every $j\in J$ and

$$\sum_{j\in J}\varphi_j\circ\psi_j=id_{\prod\limits_{j\in J}F_j}\,.\qquad\blacksquare$$

PROPOSITION 1.3.4 *Let* $(F_j \overset{\phi_j}{\to} F'_j)_{j\in J}$ *be a finite family of morphisms in* $\mathfrak{M}_E$,

$$F := \prod_{j\in J} F_j\,, \qquad F' := \prod_{j\in J} F'_j\,,$$

and for every $j \in J$ *let*

$$\varphi_j : F_j \longrightarrow F\,, \qquad \varphi'_j : F'_j \longrightarrow F'$$

be the inclusion maps. Then the diagram

$$\begin{tikzcd}
\prod\limits_{j\in J} K_i(F_j) \arrow[r, "\sum\limits_{j\in J} K_i(\varphi_j)"] \arrow[d, "\prod\limits_{j\in J} K_i(\phi_j)"'] & K_i(F) \arrow[d, "K_i\left(\prod\limits_{j\in J}\phi_j\right)"] \\
\prod\limits_{j\in J} K_i\left(F'_j\right) \arrow[r, "\sum\limits_{j\in J} K_i\left(\varphi'_j\right)"'] & K_i(F')
\end{tikzcd}$$

is commutative.

For every $j \in J$ the diagram

$$\begin{tikzcd}
F_j \arrow[r, "\varphi_j"] \arrow[d, "\phi_j"'] & F \arrow[d, "\prod\limits_{j\in J}\phi_j"] \\
F'_j \arrow[r, "\varphi'_j"'] & F'
\end{tikzcd}$$

is commutative so the diagram

$$\begin{tikzcd}
K_i(F_j) \arrow[r, "K_i(\varphi_j)"] \arrow[d, "K_i(\phi_j)"'] & K_i(F) \arrow[d, "K_i\left(\prod\limits_{j\in J}\phi_j\right)"] \\
K_1(F'_j) \arrow[r, "K_i\left(\varphi'_j\right)"'] & K_i(F')
\end{tikzcd}$$

is also commutative. For $(a_j)_{j\in J} \in \prod\limits_{j\in J} K_i(F_j)$, by the above,

$$K_i\left(\prod_{j\in J}\phi_j\right) \circ \left(\sum_{j\in J} K_i(\varphi_j)\right)(a_j)_{j\in J} = K_i\left(\prod_{j\in J}\phi_j\right)\sum_{j\in J} K_i(\varphi_j)\,a_j =$$

$$= \sum_{j\in J} K_i\left(\prod_{k\in J}\phi_k\right) K_i(\varphi_j)\, a_j = \sum_{j\in J} K_i\left(\varphi'_j\right) K_i(\phi_j)\, a_j =$$

$$= \left(\sum_{j\in J} K_i\left(\varphi'_j\right)\right)(K_i(\phi_j)\, a_j)_{j\in J} = \left(\sum_{j\in J} K_i\left(\varphi'_j\right)\right) K_i\left(\prod_{j\in J}\phi_j\right)(a_j)_{j\in J},$$

which proves the assertion. ■

PROPOSITION 1.3.5

a) If $F \overset{\varphi}{\to} G$, $G \overset{\psi}{\to} F$ is a homotopy in $\mathfrak{M}_E$ then

$$K_i(\varphi)\circ K_i(\psi) = id_{K_i(G)}\,, \qquad K_i(\psi)\circ K_i(\varphi) = id_{K_i(F)}\,.$$

b) If F and G are homotopic E-C-algebras then $K_i(F)$ and $K_i(G)$ are isomorphic.*

c) If the E-C-algebra F is null-homotopic then it is K-null.*

a) follows from the homotopy axiom (Axiom 1.2.5).

b) follows from a).

c) follows from b) and from the null-axiom (Axiom 1.2.2). ■

PROPOSITION 1.3.6 *Let*

$$0 \longrightarrow F \overset{\varphi}{\longrightarrow} G \overset{\psi}{\longrightarrow} H \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$.

a) If F (resp. H) is K-null then

$$K_i(G) \xrightarrow{K_i(\psi)} K_i(H) \qquad (\text{resp. } K_i(F) \xrightarrow{K_i(\varphi)} K_i(G))$$

is a group isomorphism.

b) If G is K-null then

$$K_i(H) \overset{\delta_i}{\longrightarrow} K_{i+1}(F)$$

is a group isomorphism.

c) If φ is K-null then the sequences

$$0 \longrightarrow K_i(G) \xrightarrow{K_i(\psi)} K_i(H) \xrightarrow{\delta_i} K_{i+1}(F) \longrightarrow 0$$

is exact.

d) If ψ is K-null then the sequences

$$0 \longrightarrow K_i(H) \xrightarrow{\delta_i} K_{i+1}(F) \xrightarrow{K_{i+1}(\varphi)} K_{i+1}(G) \longrightarrow 0$$

is exact.

e) The index maps of a split exact sequence are equal to 0.

a), b), c), and d) follow from the six-term axiom (Axiom 1.2.7).

e) follows from the six-term axiom (Axiom 1.2.7) and from the split exact axiom (Axiom 1.2.3). ■

PROPOSITION 1.3.7 *An $\mathfrak{M}_E$* **-triple** *is a triple (F_1, F_2, F_3) such that F_1 is an E-C*-algebra, F_2 is an E-ideal of F_1, and F_3 is an E-ideal of F_1 and of F_2. We denote for all $j,k \in \mathbb{N}_3$, $j<k$, by $\varphi_{j,k} : F_k \longrightarrow F_j$ the inclusion map, by $\psi_{j,k} : F_j \longrightarrow F_j/F_k$ the quotient map, and by $\delta_{j,k,i} : K_i(F_j/F_k) \longrightarrow F_k$ the index maps associated to the exact sequence in $\mathfrak{M}_E$*

$$0 \longrightarrow F_k \xrightarrow{\varphi_{j,k}} F_j \xrightarrow{\psi_{j,k}} F_j/F_k \longrightarrow 0 \, .$$

a) There is a unique morphism $F_2/F_3 \xrightarrow{\varphi_{1,2}/F_3} F_1/F_3$ in $\mathfrak{M}_E$ such that

$$\psi_{1,3} \circ \varphi_{1,2} = (\varphi_{1,2}/F_3) \circ \psi_{2,3} \, .$$

b) The diagram

$$\begin{array}{ccccccc}
K_i(F_3) & \xrightarrow{K_i(\varphi_{1,3})} & K_i(F_1) & \xrightarrow{K_i(\psi_{1,3})} & K_i(F_1/F_3) & \xrightarrow{\delta_{1,3,i}} & K_{i+1}(F_3) \\
= \uparrow & & K_i(\varphi_{1,2}) \uparrow & & \uparrow K_i(\varphi_{1,2}/F_3) & & \uparrow = \\
K_i(F_3) & \xrightarrow[K_i(\varphi_{2,3})]{} & K_i(F_2) & \xrightarrow[K_i(\psi_{2,3})]{} & K_i(F_2/F_3) & \xrightarrow[\delta_{2,3,i}]{} & K_{i+1}(F_3)
\end{array}$$

is commutative.

a) is easy to see.

b) follows from a), $\varphi_{1,2} \circ \varphi_{2,3} = \varphi_{1,3}$, and from the axiom of commutativity of the index maps (Axiom 1.2.8). ∎

THEOREM 1.3.8 (The triple theorem) *Let (F_1, F_2, F_3) be an $\mathfrak{M}_E$-triple.*

a) Assume F_2 K-null.

a_1) $\delta_{2,3,i} : K_i(F_2/F_3) \longrightarrow K_{i+1}(F_3)$ is a group isomorphism.

a_2) $\delta_{2,3,i} = \delta_{1,3,i} \circ K_i(\varphi_{1,2}/F_3)$.

a_3) $\varphi_{1,3}$ is K-null.

a_4) If we put $\Phi_i := K_i(\varphi_{1,2}/F_3) \circ (\delta_{2,3,i})^{-1}$ then

$$0 \longrightarrow K_i(F_1) \xrightarrow{K_i(\psi_{1,3})} K_i(F_1/F_3) \underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}} K_{i+1}(F_3) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(F_1) \times K_{i+1}(F_3) \longrightarrow K_i(F_1/F_3), \quad (a,b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

is a group isomorphism.

b) Assume F_1/F_3 K-null.

b_1) $\delta_{2,3,i} = 0$ and the sequence

$$0 \longrightarrow K_i(F_3) \xrightarrow{K_i(\varphi_{2,3})} K_i(F_2) \xrightarrow{K_i(\psi_{2,3})} K_i(F_2/F_3) \longrightarrow 0$$

is exact.

b_2) $K_i(\varphi_{1,3}) : K_i(F_3) \longrightarrow K_i(F_1)$ is a group isomorphism.

b_3) If we put $\Phi_i := K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2})$ then the map

$$\Psi : K_i(F_2) \longrightarrow K_i(F_3) \times K_i(F_2/F_3), \quad b \longmapsto (\Phi_i b, K_i(\psi_{2,3})b)$$

is a group isomorphism.

b$_4$) *If $\psi_{1,2}$ is K-null and if we put $\Phi'_i := K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1}$ then*

$$0 \longrightarrow K_{i+1}(F_1/F_2) \xrightarrow{\delta_{1,2,(i+1)}} K_i(F_2) \underset{\Phi'_i}{\overset{K_i(\varphi_{1,2})}{\rightleftarrows}} K_i(F_1) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(F_1) \times K_{i+1}(F_1/F_2) \longrightarrow K_i(F_2), \quad (a,b) \longmapsto \Phi'_i a + \delta_{1,2,(i+1)} b$$

is a group isomorphism.

c) *Assume F_1 K-null and denote by ψ the canonical map $F_1/F_3 \to F_1/F_2$.*

c$_1$) *$\delta_{1,2,i}$ and $\delta_{1,3,i}$ are group isomorphisms.*

c$_2$) *$K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} = \delta_{1,2,(i+1)} \circ K_{i+1}(\psi)$.*

c$_3$) *Let $\varphi : F_1/F_2 \longrightarrow F_1/F_3$ be a morphism in $\mathfrak{M}_E$ such that*

$$K_i(\psi \circ \varphi) = id_{K_i(F_1/F_2)} .$$

If we put

$$\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1}$$

then $K_i(\varphi_{2,3}) \circ \Phi_i = id_{K_i(F_2)}$. If in addition $\psi_{2,3}$ is K-null then

$$0 \longrightarrow K_{i+1}(F_2/F_3) \xrightarrow{\delta_{2,3,(i+1)}} K_i(F_3) \underset{\Phi_i}{\overset{K_i(\varphi_{2,3})}{\rightleftarrows}} K_i(F_2) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}(F_2/F_3) \times K_i(F_2) \longrightarrow K_i(F_3), \quad (a,b) \longmapsto \delta_{2,3,(i+1)} a + \Phi_i b$$

is a group isomorphism.

a_1) follows from Proposition 1.3.6 b).

a_2) follows from Proposition 1.3.7 b).

a_3) $\varphi_{1,3}$ factorizes through null and so it is K-null.

a_4) By a_2),

$$\delta_{1,3,i} \circ \Phi_i = \delta_{1,3,i} \circ K_i(\varphi_{1,2}/F_3) \circ (\delta_{2,3,i})^{-1} = \delta_{2,3,i} \circ (\delta_{2,3,i})^{-1} = id_{K_i(F_3)}$$

and this implies the assertion.

b_1) By Proposition 1.3.7 b), $\delta_{2,3,i}$ factorizes through null and so it is K-null. By the six-term axiom (Axiom 1.2.7) the sequence

$$0 \longrightarrow K_i(F_3) \xrightarrow{K_i(\varphi_{2,3})} K_i(F_2) \xrightarrow{K_i(\psi_{2,3})} K_i(F_2/F_3) \longrightarrow 0$$

is exact.

b_2) follows from Proposition 1.3.6 a).

b_3) Step 1

$$\Phi_i \circ K_i(\varphi_{2,3}) = id_{K_i(F_3)}$$

Since $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3}$,

$$\Phi_i \circ K_i(\varphi_{2,3}) = K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2}) \circ K_i(\varphi_{2,3}) =$$

$$= K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,3}) = id_{K_i(F_3)} .$$

Step 2 Ψ is injective

Let $b \in K_i(F_2)$ with $\Psi b = 0$. Then $K_i(\psi_{2,3})b = 0$ so by b_1),

$$b \in Ker\, K_i(\psi_{2,3}) = Im\, K_i(\varphi_{2,3})$$

and there is an $a \in K_i(F_3)$ with $b = K_i(\varphi_{2,3})a$. By Step 1,

$$a = \Phi_i K_i(\varphi_{2,3})a = \Phi_i b = 0 ,$$

so $b = 0$ and Ψ is injective.

Step 3 Ψ is surjective

Let $(a,c) \in K_i(F_3) \times K_i(F_2/F_3)$. Put $b' := K_i(\varphi_{2,3})a$. By b_1),

$$K_i(\psi_{2,3})b' = K_i(\psi_{2,3})K_i(\varphi_{2,3})a = 0$$

and by Step 1, $\Phi_i b' = \Phi_i K_i(\varphi_{2,3})a = a$. By b_1), there is a $b'' \in K_i(F_2)$ with $c = K_i(\psi_{2,3})b''$. By Step 1,

$$\Phi_i(b'' - K_i(\varphi_{2,3})\Phi_i b'') = \Phi_i b'' - \Phi_i K_i(\varphi_{2,3})\Phi_i b'' = \Phi_i b'' - \Phi_i b'' = 0\,.$$

Thus by b_1),

$$\Psi(b' + b'' - K_i(\varphi_{2,3})\Phi_i b'') =$$

$$= (\Phi_i b', K_i(\psi_{2,3})b'' - K_i(\psi_{2,3})K_i(\varphi_{2,3})\Phi_i b'') = (a, c)$$

and Ψ is surjective.

b_4) Since $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3}$,

$$K_i(\varphi_{1,2}) \circ \Phi'_i = K_i(\varphi_{1,2}) \circ K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1} =$$

$$= K_i(\varphi_{1,3}) \circ K_i(\varphi_{1,3})^{-1} = id_{K_i(F_1)}$$

and the assertion follows.

c_1) follows from Proposition 1.3.6 b)).

c_2) follows from the commutativity of the index maps (Axiom 1.2.8).

c_3) By c_2),

$$K_i(\varphi_{2,3}) \circ \Phi_i = K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1} =$$

$$= \delta_{1,2,(i+1)} \circ K_{i+1}(\psi) \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1} =$$

$$= \delta_{1,2,(i+1)} \circ K_{i+1}(\psi \circ \varphi) \circ (\delta_{1,2,(i+1)})^{-1} = \delta_{1,2,(i+1)} \circ (\delta_{1,2,(i+1)})^{-1} = id_{K_i(F_2)}\,.$$

The last assertion follows from the first one. ■

Remark. a) still holds with the weaker assumption that F_2 is only an E-C*-subalgebra of F_1.

1.4 Tensor Products

Throughout this section F denotes an E-C*-algebra.

DEFINITION 1.4.1 *Let G be a C*-algebra. We denote by $F \otimes G$ the spatial tensor product of F and G endowed with the structure of an E-C*-algebra by using the exterior multiplication*

$$E \times (F \otimes G) \longrightarrow F \otimes G, \quad (\alpha, x \otimes y) \longmapsto (\alpha x) \otimes y$$

([5] Proposition T.5.14 and T.5.17 Remark). *If $F \xrightarrow{\varphi} F'$ is a morphism in $\mathfrak{M}_E$ and $G \xrightarrow{\psi} G'$ a morphism in $\mathfrak{M}_{\mathbb{C}}$ then $F \otimes G \xrightarrow{\varphi \otimes \psi} F' \otimes G'$ denotes the morphism in $\mathfrak{M}_E$ defined by*

$$\varphi \otimes \psi : F \otimes G \longrightarrow F' \otimes G', \quad x \otimes y \longmapsto \varphi x \otimes \psi y \, .$$

If $(G_j)_{j \in J}$ is a family of C-algebras then we put*

$$\bigotimes_{j \in \emptyset} G_j := \mathbb{C} \, .$$

We have $F \otimes \mathbb{C} \approx F$ and $id_F \otimes id_G = id_{F \otimes G}$. If $F \xrightarrow{\varphi} F' \xrightarrow{\varphi'} F''$ are morphisms in $\mathfrak{M}_E$ and $G \xrightarrow{\psi} G' \xrightarrow{\psi'} G''$ are morphisms in $\mathfrak{M}_{\mathbb{C}}$ then

$$(\varphi \otimes \psi) \circ (\varphi' \otimes \psi') = (\varphi \circ \varphi') \otimes (\psi \circ \psi') \, .$$

If G and H are C*-algebras then

$$F \otimes (G \times H) \approx (F \otimes G) \times (F \otimes H), \qquad F \otimes (G \otimes H) \approx (F \otimes G) \otimes H \, .$$

If G is a C*-algebra and F_1, F_2 are E-C*-algebras then

$$(F_1 \times F_2) \otimes G \approx (F_1 \otimes G) \times (F_2 \otimes G) \, .$$

PROPOSITION 1.4.2 *Let G, H be C*-algebras.*

a) If $\varphi_0, \varphi_1 : G \longrightarrow H$ are homotopic C-homomorphisms then $id_F \otimes \varphi_0$ and $id_F \otimes \varphi_1$ are also homotopic.*

b) If $G \xrightarrow{\varphi} H$, $H \xrightarrow{\psi} G$ is a homotopy in $\mathfrak{M}_{\mathbb{C}}$ then

$$F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \, , \qquad F \otimes H \xrightarrow{id_F \otimes \psi} F \otimes G$$

is a homotopy in $\mathfrak{M}_E$.

c) If G is homotopic to 0 then $F \otimes G$ is also homotopic to 0 and so K-null.

a) Let $[0,1] \longrightarrow \varphi_s$ be a pointwise continuous map of C*-homomorphisms $G \to H$. Let $z \in F \odot G$. There are finite families $(x_j)_{j\in J}$ in F and $(y_j)_{j\in J}$ in G such that

$$z = \sum_{j\in J} x_j \otimes y_j \ .$$

For $s \in [0,1]$,

$$(id_F \otimes \varphi_s)z = \sum_{j\in J} x_j \otimes \varphi_s y_j$$

so the map

$$[0,1] \longrightarrow F \otimes H, \quad s \longmapsto (id_F \otimes \varphi_s)z$$

is continuous.

Let now $z \in F \otimes G$, $s_0 \in [0,1]$, and $\varepsilon > 0$. There is a $z' \in F \odot G$ such that $\|z - z'\| < \frac{\varepsilon}{3}$. By the above, there is a $\delta > 0$ such that

$$\left\|(id_F \otimes \varphi_s)z' - (id_F \otimes \varphi_{s_0})z'\right\| < \frac{\varepsilon}{3}$$

for all $s \in [0,1]$, $|s - s_0| < \delta$. It follows

$$\left\|(id_F \otimes \varphi_s)z - (id_F \otimes \varphi_{s_0})z\right\| \le \left\|(id_F \otimes \varphi_s)(z - z')\right\| +$$

$$+ \left\|(id_F \otimes \varphi_s)z' - (id_F \otimes \varphi_{s_0})z'\right\| + \left\|(id_F \otimes \varphi_{s_0})(z - z')\right\| < \varepsilon\,,$$

which proves the assertion.

b) follows from a).

c) follows from b) and Proposition 1.3.5 c)). ■

PROPOSITION 1.4.3 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \underset{\xleftarrow[\lambda]{}}{\xrightarrow{\psi}} G_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_{\mathbf{C}}$.

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \underset{\xleftarrow[id_F \otimes \lambda]{}}{\xrightarrow{id_F \otimes \psi}} F \otimes G_3 \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F \otimes G_1) \xrightarrow{K_i(id_F \otimes \varphi)} K_i(F \otimes G_2) \underset{\xleftarrow[K_i(id_F \otimes \lambda)]{}}{\xrightarrow{K_i(id_F \otimes \psi)}} K_i(F \otimes G_3) \longrightarrow 0$$

is split exact and the map

$$K_i(F \otimes G_1) \times K_i(F \otimes G_3) \longrightarrow K_i(F \otimes G_2)\,,$$

$$(a,b) \longmapsto K_i(id_F \otimes \varphi)a + K_i(id_F \otimes \lambda)b$$

is a group isomorphism.

a) By [5] Corollary T.5.19, $id_F \otimes \varphi$ is injective. We have

$$(id_F \otimes \psi) \circ (id_F \otimes \lambda) = id_F \otimes (\psi \circ \lambda) = id_F \otimes id_{G_3} = id_{F \otimes G_3}\,,$$

$$(id_F \otimes \psi) \circ (id_F \circ \varphi) = id_F \otimes (\psi \circ \varphi) = 0\,,$$

so

$$Im\,(id_F \otimes \varphi) \subset Ker\,(id_F \otimes \psi)\ .$$

Let $z \in (F \odot G_2) \cap Ker\,(id_F \otimes \psi)$. There is a linearly independent finite family $(x_j)_{j \in J}$ in F and a family $(y_j)_{j \in J}$ in G_2 such that

$$z = \sum_{j \in J} x_j \otimes y_j\ .$$

From

$$0 = (id_F \otimes \psi)z = \sum_{j \in J} x_j \otimes \psi y_j$$

we get $\psi y_j = 0$ for all $j \in J$. Thus for every $j \in J$ there is a $y'_j \in G_1$ with $\varphi y'_j = y_j$. It follows

$$z = \sum_{j \in J} x_j \otimes \varphi y'_j = (id_F \otimes \varphi) \sum_{j \in J} x_j \otimes y'_j \in Im\,(id_F \otimes \varphi)\ .$$

Let $z \in Ker\,(id_F \otimes \psi)$. Then

$$(id_F \otimes (\lambda \circ \psi))z = (id_F \otimes \lambda)(id_F \otimes \psi)z = 0\ .$$

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $F \odot G_2$ converging to z. For $n \in \mathbb{N}$, by the above,

$$(id_F \otimes \psi)(z_n - (id_F \otimes (\lambda \circ \psi))z_n) =$$

$$= (id_F \otimes \psi)z_n - (id_F \otimes \psi)(id_F \otimes \lambda)(id_F \otimes \psi)z_n =$$

$$= (id_F \otimes \psi)z_n - (id_F \otimes \psi)z_n = 0\,,$$

$$z_n - (id_F \otimes (\lambda \circ \psi))z_n \in Im\,(id_F \otimes \psi)\;.$$

Since $Im\,(id_F \otimes \varphi)$ is closed,

$$z = z - (id_F \otimes (\lambda \circ \psi))z = \lim_{n\to\infty}(z_n - (id_F \otimes (\lambda \circ \psi))z_n) \in Im\,(id_F \otimes \varphi)\,,$$

which proves the Proposition.

b) follows from a) and the split exact axiom (Axiom 1.2.3). ■

DEFINITION 1.4.4

We denote for every C-algebra G by $\tilde{G}$ its unitization* (see e.g. [4] Exercise 1.3) *and by*

$$0 \longrightarrow G \xrightarrow{\iota_G} \tilde{G} \underset{\overleftarrow{\lambda_G}}{\overset{\pi_G}{\longrightarrow}} \mathbb{C} \longrightarrow 0$$

its associated split exact sequence. If G and H are C-algebras and $\varphi : G \longrightarrow H$ is a C*-homomorphism then $\tilde{\varphi} : \tilde{G} \longrightarrow \tilde{H}$ denotes the unitization of φ.*

COROLLARY 1.4.5 *Let G be a C*-algebra.*

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \iota_G} F \otimes \tilde{G} \underset{\overleftarrow{id_F \otimes \lambda_G}}{\overset{id_F \otimes \pi_G}{\longrightarrow}} F \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(id_F \otimes \iota_G)} K_i\left(F \otimes \tilde{G}\right) \underset{\overleftarrow{K_i(id_F \otimes \lambda_G)}}{\overset{K_i(id_F \otimes \pi_G)}{\longrightarrow}} K_i(F) \longrightarrow 0$$

is split exact and the map

$$K_i(F) \times K_i(F \otimes G) \longrightarrow K_i\left(F \otimes \tilde{G}\right)\,,$$

$$(a,b) \longmapsto K_i\,(id_F \otimes \lambda_G)\,a + K_i\,(id_F \otimes \iota_G)\,b$$

is a group isomorphism.

c) Let $F \stackrel{\varphi}{\to} F'$ *be a morphism in* $\mathfrak{M}_E$ *and* $G \stackrel{\psi}{\to} G'$ *a morphism in* $\mathfrak{M}_{\mathbf{C}}$*. If we identify the isomorphic groups of b) then*

$$K_i(\varphi \otimes \tilde{\psi}) : K_i\left(F \otimes \tilde{G}\right) \longrightarrow K_i\left(F' \otimes \tilde{G}'\right),$$

$$(a,b) \longmapsto (K_i(\varphi)\, a, K_i(\varphi \otimes \psi)\, b)$$

is a group isomorphism.

d) Let $\varphi : G \longrightarrow G'$ *be a morphism in* $\mathfrak{M}_{\mathbf{C}}$*. If we denote by* Ψ_i *and* Ψ'_i *the group isomorphisms of b) associated to* G *and* G'*, respectively, then*

$$K_i(id_F \otimes \tilde{\varphi}) \circ \Psi_i = \Psi'_i \circ \left(id_{K_i(F)} \times K_i(id_F \otimes \varphi)\right) .$$

a) and b) follow from Proposition 1.4.3 a),b).

c) follows from b) and the commutativity of the following diagram:

$$\begin{tikzcd}
F \otimes G \arrow[r, "id_F \otimes \iota_G"] \arrow[d, "\varphi \otimes \psi"'] & F \otimes \tilde{G} \arrow[d, "\varphi \otimes \tilde{\psi}"'] & F \otimes \mathbf{C} \arrow[l, "id_F \otimes \lambda_G"'] \arrow[d, "\varphi \otimes id_{\mathbf{C}}"] \\
F' \otimes G' \arrow[r, "id_{F'} \otimes \iota_{G'}"'] & F' \otimes \tilde{G}' & F' \otimes \mathbf{C} \arrow[l, "id_{F'} \otimes \lambda_{G'}"]
\end{tikzcd} .$$

d) For $(a,b) \in K_i(F) \times K_i(F \otimes G)$, since $\tilde{\varphi} \circ \lambda_G = \lambda_{G'}$ and $\iota_{G'} \circ \varphi = \tilde{\varphi} \circ \iota_G$,

$$K_i(id_F \otimes \tilde{\varphi})\Psi_i(a,b) = K_i(id_F \otimes \tilde{\varphi})(K_i(id_F \otimes \lambda_G)\, a + K_i(id_F \otimes \iota_G)\, b) =$$

$$= K_i(id_F \otimes \tilde{\varphi}) K_i(id_F \otimes \lambda_G)\, a + K_i(id_F \otimes \tilde{\varphi}) K_i(id_F \otimes \iota_G)\, b =$$

$$= K_i(id_F \otimes (\tilde{\varphi} \circ \lambda_G))\, a + K_i(id_F \otimes (\tilde{\varphi} \circ \iota_G))\, b =$$

$$= K_i(id_F \otimes \lambda_{G'})\, a + K_i(id_F \otimes (\iota_{G'} \circ \varphi))\, b =$$

$$= K_i(id_F \otimes \lambda_{G'})\, a + K_i(id_F \otimes \iota_{G'}) K_i(id_F \otimes \varphi)\, b =$$

$$= \Psi'_i(a, K_i(id_F \otimes \varphi)\, b) = \Psi'_i(id_{K_i(F)} \times K_i(id_F \otimes \varphi))(a,b),$$

so

$$K_i(id_F \otimes \tilde{\varphi}) \circ \Psi_i = \Psi'_i \circ \left(id_{K_i(F)} \times K_i(id_F \otimes \varphi)\right) . \qquad \blacksquare$$

PROPOSITION 1.4.6 *If $(G_j)_{j\in J}$ is a finite family of C*-algebras then*

$$K_i\left(F\otimes\left(\bigotimes_{j\in J}\tilde{G}_j\right)\right)\approx\prod_{I\subset J}K_i\left(F\otimes\left(\bigotimes_{j\in I}G_j\right)\right).$$

We prove the assertion by induction with respect to $Card\,J$. The assertion is trivial for $Card\,J=0$ (Definition 1.4.1 and Null-axiom (Axiom 1.4.6)). Let $j_0\in J$, $J':=J\setminus\{\mathrm{j}_0\}$, and assume the assertion holds for J'. By Corollary 1.4.5 b),

$$K_i\left(F\otimes\left(\bigotimes_{j\in J}\tilde{G}_j\right)\right)\approx K_i\left(\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\otimes\tilde{G}_{j_0}\right)\approx$$

$$\approx K_i\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\times K_i\left(\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\otimes G_{j_0}\right)\approx$$

$$\approx K_i\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\times K_i\left((F\otimes G_{j_0})\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\approx$$

$$\approx\prod_{I\subset J'}K_i\left(F\otimes\left(\bigotimes_{j\in I}G_j\right)\right)\times\prod_{I\subset J'}K_i\left(F\otimes\left(\bigotimes_{j\in I\cup\{\mathrm{j}_0\}}G_j\right)\right)\approx$$

$$\approx\prod_{I\subset J}K_i\left(F\otimes\left(\bigotimes_{j\in I}G_j\right)\right).\qquad\blacksquare$$

COROLLARY 1.4.7 *If G is a C*-algebra then for all $n\in\mathbb{N}^*$*

$$K_i\left(F\otimes\left(\bigotimes_{j\in\mathbb{N}_n}\tilde{G}\right)\right)\approx\prod_{k=0}^{n}K_i\left(F\otimes\left(\bigotimes_{j\in\mathbb{N}_k}G\right)\right)^{\binom{n}{k}}.\qquad\blacksquare$$

PROPOSITION 1.4.8 *Let G be a C*-algebra and*

$$0\longrightarrow F_1\xrightarrow{\varphi}F_2\underset{\xleftarrow[\lambda]{}}{\xrightarrow{\psi}}F_3\longrightarrow 0$$

a split exact sequence in $\mathfrak{M}_E$.

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\varphi \otimes id_G} F_2 \otimes G \underset{\overleftarrow{\lambda \otimes id_G}}{\xrightarrow{\psi \otimes id_G}} F_3 \otimes G \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F_1 \otimes G) \xrightarrow{K_i(\varphi \otimes id_G)} K_i(F_2 \otimes G) \underset{\overleftarrow{K_i(\lambda \otimes id_G)}}{\xrightarrow{K_i(\psi \otimes id_G)}} K_i(F_3 \otimes G) \longrightarrow 0$$

is split exact and the map

$$K_i(F_1 \otimes G) \times K_i(F_3 \otimes G) \longrightarrow K_i(F_2 \otimes G),$$

$$(a,b) \longmapsto K_i(\varphi \otimes id_G)a + K_i(\lambda \otimes id_G)b$$

is a group isomorphism.

The proof is similar to the proof of Proposition 1.4.3. ■

PROPOSITION 1.4.9 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$. *If* F *or* G_3 *is nuclear then the sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightarrow{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0$$

is exact and so

$$\frac{F \otimes G_2}{F \otimes G_1} \approx F \otimes \frac{G_2}{G_1}.$$

[5] Theorem T.6.26. ■

PROPOSITION 1.4.10 *Let* G *be a C*-algebra and*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

an exact sequence in $\mathfrak{M}_E$. *If* F_3 *or* G *is nuclear then*

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\phi_1 \otimes id_G} F_2 \otimes G \xrightarrow{\phi_2 \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is exact.

[5] Theorem T.6.26. ■

DEFINITION 1.4.11 *Let*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$ and G a C-algebra. If δ_i denotes the index maps associated to the above exact sequence in $\mathfrak{M}_E$ and if the sequence in $\mathfrak{M}_E$*

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\phi_1 \otimes id_G} F_1 \otimes G \xrightarrow{\phi_2 \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is exact (e.g. F_3 or G is nuclear ([5] T.6.26)) *then we denote by $\delta_{G,i}$ the index maps associated to this last exact sequence in $\mathfrak{M}_E$.*

In this case the six-term sequence

$$\begin{array}{ccccc} K_0(F_1 \otimes G) & \xrightarrow{K_0(\phi_1 \otimes id_G)} & K_0(F_2 \otimes G) & \xrightarrow{K_0(\phi_2 \otimes id_G)} & K_0(F_3 \otimes G) \\ \delta_{G,1} \uparrow & & & & \downarrow \delta_{G,0} \\ K_1(F_3 \otimes G) & \xleftarrow[K_1(\phi_2 \otimes id_G)]{} & K_1(F_2 \otimes G) & \xleftarrow[K_1(\phi_1 \otimes id_G)]{} & K_1(F_1 \otimes G) \end{array}$$

is exact (by the six-term axiom (Axiom 1.2.7)).

COROLLARY 1.4.12 *Let G be a unital C*-algebra,*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

an exact sequence in $\mathfrak{M}_E$, and δ_i its index maps. We assume that F_3 or G is nuclear and put for every $j \in \{1,2,3\}$

$$\varphi_j : F_j \longrightarrow F_j \otimes G, \quad x \longmapsto x \otimes 1_G .$$

Then $\delta_{G,i} \circ K_i(\varphi_3) = K_{i+1}(\varphi_1) \circ \delta_i$.

The diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{\phi_1} & F_2 & \xrightarrow{\phi_2} & F_3 \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \downarrow \varphi_3 \\ F_1 \otimes G & \xrightarrow[\phi_1 \otimes id_G]{} & F_2 \otimes G & \xrightarrow[\phi_2 \otimes id_G]{} & F_3 \otimes G \end{array}$$

is commutative and the assertion follows from Proposition 1.4.10 and the commutativity of the index maps (Axiom 1.2.8). ■

1.5 The Class Υ

Throughout this section F denotes an E-C*-algebra.

DEFINITION 1.5.1 *Let Υ be the class of those C*-algebras G for which there are $p(G), q(G) \in \mathbb{N}^*$ and group isomorphisms*

$$\Phi_{i,G,F} : K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} \longrightarrow K_i(F \otimes G)$$

such that for every morphism $F \xrightarrow{\phi} F'$ in $\mathfrak{M}_E$ the diagram

$$\begin{array}{ccc} K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} & \xrightarrow{\Phi_{i,G,F}} & K_i(F \otimes G) \\ {\scriptstyle K_i(\phi)^{p(G)} \times K_{i+1}(\phi)^{q(G)}} \downarrow & & \downarrow {\scriptstyle K_i(\phi \otimes id_G)} \\ K_i(F')^{p(G)} \times K_{i+1}(F')^{q(G)} & \xrightarrow[\Phi_{i,G,F'}]{} & K_i(F' \otimes G) \end{array}$$

is commutative. We denote by $\vec{G}$ the class of group isomorphisms

$$\Phi_{i,G,F} : K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} \longrightarrow K_i(F \otimes G)$$

having the above property. A C-algebra G is called* **Υ-null** *if $G \in \Upsilon$ and $p(G) = q(G) = 0$.*

If G is Υ-null or if F is K-null and $G \in \Upsilon$ then $F \otimes G$ is K-null. In general we shall use $\Phi_{i,G,F}$ without writing $\{\Phi_{\mathrm{i,G,F}}\} \in \vec{G}$.

PROPOSITION 1.5.2 *Let $p.q \in \mathbb{N}^*$ and let Λ be the class of group isomorphisms*

$$\Lambda_{i,F} : K_i(F)^p \times K_{i+1}(F)^q \longrightarrow K_i(F)^p \times K_{i+1}(F)^q$$

such that for all morphisms $F \xrightarrow{\phi} F'$ in $\mathfrak{M}_E$ the diagram

$$\begin{array}{ccc} K_i(F)^p \times K_{i+1}(F)^q & \xrightarrow{\Lambda_{i,F}} & K_i(F)^p \times K_{i+1}(F)^q \\ {\scriptstyle K_i(\phi)^p \times K_{i+1}(\phi)^q} \downarrow & & \downarrow {\scriptstyle K_i(\phi)^p \times K_{i+1}(\phi)^q} \\ K_i(F')^p \times K_{i+1}(F')^q & \xrightarrow[\Lambda_{i,F'}]{} & K_i(\phi)^p \times K_{i+1}(\phi)^q \end{array}$$

is commutative. Let $G \in \Upsilon$ with $p(G) = p$, $q(G) = q$, and let $\{\Phi_{i,G,F}\} \in \vec{G}$.

a) *If* $\Lambda_{i,F} \in \Lambda$ *and if we put*

$$\Phi'_{i,G,F} := \Phi_{i,G,F} \circ \Lambda_{i,F} : K_i(\phi)^p \times K_{i+1}(\phi)^q \longrightarrow K_i(F \otimes G)$$

then $\{\Phi'_{i,G,F}\} \in \vec{G}$.

b) *If* $\{\Phi'_{i,G,F}\} \in \vec{G}$ *and if we put*

$$\Lambda_{i,F} := \Phi^{-1}_{i,G,F} \circ \Phi'_{i,G,F} : K_i(\phi)^p \times K_{i+1}(\phi)^q \longrightarrow K_i(\phi)^p \times K_{i+1}(\phi)^q$$

then $\{\Lambda_{i,F}\} \in \Lambda$.

c) *If* $\{\Lambda_{i,F}\}, \{\Lambda'_{\mathrm{i,F}}\} \in \Lambda$ *then* $\{\Lambda_{\mathrm{i,F}} \circ \Lambda'_{\mathrm{i,F}}\} \in \Lambda$, $\{\Lambda^{-1}_{\mathrm{i,F}}\} \in \Lambda$. ■

DEFINITION 1.5.3 *We denote for every nuclear* $G \in \Upsilon$ *by* G_Υ *the class of exact sequences in* $\mathfrak{M}_E$

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

such that if δ_i *denote its index maps then the diagram*

$$\begin{array}{ccc} K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\Phi_{i,G,F_3}} & K_i(F_3 \otimes G) \\ {\scriptstyle \delta_i^{p(G)} \times \delta_{i+1}^{q(G)}} \Big\downarrow & & \Big\downarrow {\scriptstyle \delta_{G,i}} \\ K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)} & \xrightarrow[\Phi_{(i+1),G,F_1}]{} & K_{i+1}(F_1 \otimes G) \end{array}$$

is commutative.

If G is Υ-null then every exact sequence in $\mathfrak{M}_E$ belongs to G_Υ.

PROPOSITION 1.5.4

a) 0 *is* Υ*-null.*

b) $\mathbb{C} \in \Upsilon$, $p(\mathbb{C}) = 1$, $q(\mathbb{C}) = 0$, $\Phi_{i,\mathbb{C},F} = K_i(\phi_{\mathbb{C},F})$, *where*

$$\phi_{\mathbb{C},F} : F \longrightarrow F \times \mathbb{C}, \quad x \longmapsto x \otimes 1_{\mathbb{C}}\ .$$

Every exact sequence in $\mathfrak{M}_E$ *belongs to* $\mathbb{C}_\Upsilon$.

c) Let $G \xrightarrow{\varphi} G'$, $G' \xrightarrow{\psi} G$ be a homotopy in $\mathfrak{M}_{\mathbf{C}}$. If $G \in \Upsilon$ then

$$G' \in \Upsilon\,, \qquad p(G') = p(G)\,, \qquad q(G') = q(G)\,,$$

$$\Phi_{i,G',F} = K_i\,(id_F \otimes \varphi) \circ \Phi_{i,G,F}\ .$$

If in addition G and G' are nuclear then $G_\Upsilon = G'_\Upsilon$.

d) If G is null-homotopic then G is Υ-null.

a) By the null-axiom (Axiom 1.2.2), 0 is Υ-null.

b) The first assertion is easy to see. The second one follows from the commutativity of the index maps (Axiom 1.2.8).

c) By Proposition 1.4.2 b),

$$F \otimes G \overset{id_F \otimes \varphi}{\longrightarrow} F \otimes G'\,, \qquad F \otimes G' \overset{id_F \otimes \psi}{\longrightarrow} F \otimes G$$

is a homotopy in $\mathfrak{M}_E$. By Proposition 1.3.7 a),

$$K_i\,(id_F \otimes \varphi) : K_i\,(F \otimes G) \longrightarrow K_i\left(F \otimes G'\right)\,,$$

$$K_i\,(id_F \otimes \psi) : K_i\left(F \otimes G'\right) \longrightarrow K_i\,(F \otimes G)$$

are group isomorphisms and $K_i\,(id_F \otimes \psi) = K_i\,(id_F \otimes \varphi)^{-1}$. Thus

$$K_i\,(id_F \otimes \varphi) \circ \Phi_{i,G,F} : K_i\,(F)^{p(G)} \times K_{i+1}\,(F)^{q(G)} \longrightarrow K_i\left(F \otimes G'\right)$$

is a group isomorphism. If $F \xrightarrow{\phi} F'$ is a morphism in $\mathfrak{M}_E$ then the diagram

$$\begin{array}{ccccc}
K_i\,(F)^{p(G)} \times K_{i+1}\,(F)^{q(G)} & \xrightarrow{\Phi_{i,G,F}} & K_i\,(F \otimes G) & \xrightarrow{K_i(id_F \otimes \varphi)} & K_i\,(F \otimes G') \\
\Big\downarrow {\scriptstyle K_i(\phi)^{p(G)} \times K_{i+1}(\phi)^{q(G)}} & & \Big\downarrow {\scriptstyle K_i(\phi \otimes id_G)} & & {\scriptstyle K_i\left(\phi \otimes id_{G'}\right)} \Big\downarrow \\
K_i\,(F')^{p(G)} \times K_{i+1}\,(F')^{q(G)} & \xrightarrow[\Phi_{i,G,F'}]{} & K_i\,(F' \otimes G) & \xrightarrow[K_i\left(id_{F'} \otimes \varphi\right)]{} & K_i\,(F' \otimes G')
\end{array}$$

is commutative and the first assertion follows.

Assume now that G and G' are nuclear, let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in G_\Upsilon\,,$$

and let δ_i be its associated index maps. By the commutativity of the index maps (Axiom 1.2.8 a)) the diagram

$$\begin{array}{ccc}
K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\delta_i^{p(G)} \times \delta_{i+1}^{q(G)}} & K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)} \\
\Phi_{i,G,F_3} \downarrow & & \downarrow \Phi_{(i+1),G,F_1} \\
K_i(F_3 \otimes G) & \xrightarrow[\delta_{G,i}]{} & K_{i+1}(F_1 \otimes G) \\
K_i(id_{F_3} \otimes \varphi) \downarrow & & \downarrow K_{i+1}(id_{F_1} \otimes \varphi) \\
K_i(F_3 \otimes G') & \xrightarrow[\delta_{G',i}]{} & K_{i+1}(F_1 \otimes G')
\end{array}$$

is commutative. Since the maps of the columns are group isomorphisms, it follows by the above, that the diagram

$$\begin{array}{ccc}
K_i(F_3)^{p(G')} \times K_{i+1}(F_3)^{q(G')} & \xrightarrow{\delta_i^{p(G')} \times \delta_{i+1}^{q(G')}} & K_{i+1}(F_1)^{p(G')} \times K_i(F_1)^{q(G')} \\
\Phi_{i,G',F_3} \downarrow & & \downarrow \Phi_{(i+1),G',F_1} \\
K_i(F_3 \otimes G') & \xrightarrow[\delta_{G',i}]{} & K_{i+1}(F_1 \otimes G')
\end{array}$$

is also commutative.

d) follows from a) and c). ■

PROPOSITION 1.5.5 *Let G be a nuclear C*-algebra belonging to Υ.*

a) Every split exact sequence in $\mathfrak{M}_E$ belongs to G_Υ.

b) Every exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

for which F_1 or F_3 is homotopic to 0 belongs to G_Υ .

a) Let

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \underset{\underset{\lambda}{\longleftarrow}}{\overset{\psi}{\longrightarrow}} F_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_E$ and let δ_i be its index maps. By Proposition 1.4.8 a),

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\varphi\otimes id_G} F_2 \otimes G \underset{\lambda\otimes id_G}{\overset{\psi\otimes id_G}{\rightleftarrows}} F_3 \otimes G \longrightarrow 0$$

is split exact and so by Proposition 1.3.1, $\delta_i = \delta_{G,i} = 0$.

b) By Proposition 1.4.2 c), $F_1 \otimes G$ or $F_3 \otimes G$ is null-homotopic and so K-null. Thus by the six-term axiom (Axiom 1.2.7), $\delta_i = \delta_{G,i} = 0$, where δ_i denote the index maps associated to

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0\,.$$ ■

PROPOSITION 1.5.6 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$ such that G_3 is nuclear.

a) Assume G_1 is Υ-null.

a_1) $K_i\,(id_F \otimes \psi) : K_i\,(F \otimes G_2) \longrightarrow K_i\,(F \otimes G_3)$ is a group isomorphism.

a_2) If $G_2 \in \Upsilon$ or $G_3 \in \Upsilon$ then

$$G_2, G_3 \in \Upsilon\,, \qquad p(G_2) = p(G_3)\,, \qquad q(G_2) = q(G_3)\,,$$

$$\Phi_{i,G_3,F} = K_i\,(id_F \otimes \psi) \circ \Phi_{i,G_2,F}\,.$$

If in addition G_2 is nuclear then $(G_2)_\Upsilon = (G_3)_\Upsilon$.

b) Assume G_2 is Υ-null and let δ_i^F denote the index maps of the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F\otimes\varphi} F \otimes G_2 \xrightarrow{id_F\otimes\psi} F \otimes G_3 \longrightarrow 0\,.$$

b_1) $\delta_i^F : K_i\,(F \otimes G_3) \longrightarrow K_{i+1}\,(F \otimes G_1)$ is a group isomorphism.

b_2) If $G_1 \in \Upsilon$ or $G_3 \in \Upsilon$ then

$$G_1, G_3 \in \Upsilon\,, \qquad p(G_1) = q(G_3)\,, \qquad q(G_1) = p(G_3)\,,$$

$$\Phi_{i,G_3,F} = \Phi_{(i+1),G_1,F} \circ \delta_i^F\,.$$

c) Assume G_3 is Υ-null.

c_1) $K_i(id_F \otimes \varphi) : K_i(F \otimes G_1) \longrightarrow K_i(F \otimes G_2)$ is a group isomorphism.

c_2) If $G_1 \in \Upsilon$ or $G_2 \in \Upsilon$ then

$$G_1, G_2 \in \Upsilon, \qquad p(G_1) = p(G_2), \qquad q(G_1) = q(G_2),$$

$$\Phi_{i,G_2,F} = K_i(id_F \otimes \varphi) \circ \Phi_{i,G_1,F} .$$

If in addition G_1 and G_2 are nuclear then $(G_1)_\Upsilon = (G_2)_\Upsilon$.

By Proposition 1.4.9 a), the sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightarrow{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0$$

is exact. If G_j is Υ-null then $F \otimes G_j$ is K-null so $a_1), b_1), c_1)$ follow from Proposition 1.3.6 a),b).

a_2) By a_1), it is easy to see that

$$G_2, G_3 \in \Upsilon, \qquad p(G_2) = p(G_3), \qquad q(G_2) = q(G_3),$$

$$\Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ \Phi_{i,G_2,F} .$$

Assume now G_2 nuclear. Let

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

belong to $(G_2)_\Upsilon$ or $(G_3)_\Upsilon$ and let δ_i be its associated index maps. Consider the diagram

$$\begin{tikzcd}
K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} \arrow[r, "\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}"] \arrow[d, "\Phi_{i,G_2,F_3}"'] & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)} \arrow[d, "\Phi_{(i+1),G_2,F_1}"] \\
K_i(F_3 \otimes G_2) \arrow[r, "\delta_{G_2,i}"] \arrow[d, "K_i(id_{F_3} \otimes \psi)"'] & K_{i+1}(F_1 \otimes G_2) \arrow[d, "K_{i+1}(id_{F_1} \otimes \psi)"] \\
K_i(F_3 \otimes G_3) \arrow[r, "\delta_{G_3,i}"'] & K_{i+1}(F_1 \otimes G_3) \\
K_i(F_3)^{p(G_3)} \times K_{i+1}(F_3)^{q(G_3)} \arrow[r, "\delta_i^{p(G_3)} \times \delta_{i+1}^{q(G_3)}"'] \arrow[u, "\Phi_{i,G_3,F_3}"] & K_{i+1}(F_1)^{p(G_3)} \times K_i(F_1)^{q(G_3)} \arrow[u, "\Phi_{(i+1),G_3,F_1}"']
\end{tikzcd}$$

Its upper part or lower part is commutative and the maps of the columns are group isomorphisms. It follows, by the above, that the diagram is commutative. Thus $(G_2)_\Upsilon = (G_3)_\Upsilon$.

b_2) Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. Then the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_i(F\otimes G_1) & \xrightarrow{K_i(id_F\otimes\varphi)} & K_i(F\otimes G_2) & \xrightarrow{K_i(id_F\otimes\psi)} & \\
 & & \Big\downarrow {\scriptstyle K_i(\phi\otimes id_{G_1})} & & \Big\downarrow {\scriptstyle K_i(\phi\otimes id_{G_2})} & & \\
0 & \longrightarrow & K_i(F'\otimes G_1) & \xrightarrow[K_i(id_{F'}\otimes\varphi)]{} & K_i(F'\otimes G_2) & \xrightarrow[K_i(id_{F'}\otimes\psi)]{} &
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{K_i(id_F\otimes\varphi)} & K_i(F\otimes G_2) & \xrightarrow{K_i(id_F\otimes\psi)} & K_i(F\otimes G_3) & \longrightarrow & 0 \\
 & \Big\downarrow {\scriptstyle K_i(\phi\otimes id_{G_2})} & & \Big\downarrow {\scriptstyle K_i(\phi\otimes id_{G_3})} & & \\
\xrightarrow[K_i(id_{F'}\otimes\varphi)]{} & K_i(F'\otimes G_2) & \xrightarrow[K_i(id_{F'}\otimes\psi)]{} & K_i(F'\otimes G_3) & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8), the diagram

$$\begin{array}{ccc}
K_i(F\otimes G_3) & \xrightarrow{\delta_i^F} & K_{i+1}(F\otimes G_1) \\
\Big\downarrow {\scriptstyle K_i(\phi\otimes id_{G_3})} & & \Big\downarrow {\scriptstyle K_{i+1}(\phi\otimes id_{G_1})} \\
K_i(F'\otimes G_3) & \xrightarrow[\delta_i^{F'}]{} & K_{i+1}(F'\otimes G_1)
\end{array}$$

is commutative. By b_1),

$$G_1, G_3 \in \Upsilon, \qquad p(G_1) = q(G_3), \qquad q(G_1) = p(G_3),$$

$$\Phi_{i,G_3,F} = \Phi_{(i+1),G_1,F} \circ \delta_i^F .$$

c_2) The proof is similar to the proof of a_2). ■

PROPOSITION 1.5.7 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \underset{\lambda}{\overset{\psi}{\rightleftarrows}} G_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_{\mathbf{C}}$.

a) If $G_1, G_3 \in \Upsilon$ then

$$G_2 \in \Upsilon, \qquad p(G_2) = p(G_1) + p(G_3), \qquad q(G_2) = q(G_1) + q(G_3),$$

$$\Phi_{i,G_2,F} = (K_i(id_F \otimes \varphi) \times K_i(id_F \otimes \lambda)) \circ (\Phi_{i,G_1,F} \times \Phi_{i,G_3,F}).$$

b) If in addition G_1,G_2,and G_3 are nuclear then $(G_1)_\Upsilon \cap (G_3)_\Upsilon \subset (G_2)_\Upsilon$.

a) By Proposition 1.4.3 b), the sequence

$$0 \longrightarrow K_i(F \otimes G_1) \xrightarrow{K_i(id_F \otimes \varphi)} K_i(F \otimes G_2) \underset{\xleftarrow[K_i(id_F \otimes \lambda)]{}}{\xrightarrow{K_i(id_F \otimes \psi)}} K_i(F \otimes G_3) \longrightarrow 0$$

is split exact. Thus the maps

$$\left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)}\right) \times \left(K_i(F)^{p(G_3)} \times K_{i+1}(F)^{q(G_3)}\right)$$

$$\xrightarrow{\Phi_{i,G_1,F} \times \Phi_{i,G_3,F}} K_i(F \otimes G_1) \times K_i(F \otimes G_3) \xrightarrow{K_i(id_F \otimes \varphi) \times K_i(id_F \otimes \lambda)} K_i(F \otimes G_2)$$

are group isomorphisms.

Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. Since the diagram with split exact rows

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \underset{\xleftarrow[id_F \otimes \lambda]{}}{\xrightarrow{id_F \otimes \psi}} F \otimes G_3 \longrightarrow 0,$$

$$0 \longrightarrow F' \otimes G_1 \xrightarrow{id_{F'} \otimes \varphi} F' \otimes G_2 \underset{\xleftarrow[id_{F'} \otimes \lambda]{}}{\xrightarrow{id_{F'} \otimes \psi}} F' \otimes G_3 \longrightarrow 0,$$

(Proposition 1.4.3 a)) and with columns $\phi \otimes id_{G_1}$, $\phi \otimes id_{G_2}$, and $\phi \otimes id_{G_3}$ is commutative, the assertion follows from Proposition 1.4.3 b).

b) Let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in (G_1)_\Upsilon \cap (G_3)_\Upsilon$$

and let δ_i be its associated index maps. Consider the diagram (by a))

$$\begin{array}{ccc}
K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)} \\
\Phi_{i,G_1,F_3} \times \Phi_{i,G_3,F_3} \Big\downarrow & & \Phi_{(i+1),G_1,F_1} \times \Phi_{(i+1),G_3,F_1} \Big\downarrow \\
K_i(F_3 \otimes G_1) \times K_i(F_3 \otimes G_3) & \xrightarrow[\delta_{G_1,i} \times \delta_{G_3,i}]{} & K_{i+1}(F_1 \otimes G_1) \times K_{i+1}(F_1 \otimes G_3) \\
A \Big\downarrow & & K_{i+1}\left(id_{F_1} \otimes \varphi\right) \times K_{i+1}\left(id_{F_1} \otimes \lambda\right) \Big\downarrow \\
K_i(F_3 \otimes G_2) & \xrightarrow[\delta_{G_2,i}]{} & K_{i+1}(F_1 \otimes G_2) \\
\Phi_{i,G_2,F_3} \Big\uparrow & & \Big\uparrow \Phi_{(i+1),G_2,F_1} \\
K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)} ,
\end{array}$$

where

$$A := K_i\left(id_{F_3} \otimes \varphi\right) \times K_i\left(id_{F_3} \otimes \lambda\right) .$$

Its upper part is commutative and the maps of the columns are group isomorphisms. It follows that the lower part of the diagram is also commutative. ■

COROLLARY 1.5.8 *If $G \in \Upsilon$ then $\tilde{G} \in \Upsilon$, $p(\tilde{G}) = p(G) + 1$, $q(\tilde{G}) = q(G)$. If in addition G and $\tilde{G}$ are nuclear then $G_\Upsilon \subset \tilde{G}_\Upsilon$.* ■

PROPOSITION 1.5.9 *Let $(G_j)_{j \in J}$ be a finite family in Υ.*

a)

$$G := \prod_{j \in J} G_j \in \Upsilon, \qquad p(G) = \sum_{j \in J} p(G_j), \qquad q(G) = \sum_{j \in J} q(G_j),$$

$$\Phi_{i,G,F} = \left(\prod_{j \in J} \Phi_{i,G_j,F} \right) \circ \Phi_{(F \otimes G_j)_{j \in J},i} .$$

In particular if G_j is Υ-null for every $j \in J$ then G is Υ-null.

b) If in addition G and all G_j, $j \in J$, are nuclear then

$$\bigcap_{j \in J} (G_j)_\Upsilon \subset G_\Upsilon .$$

c) $\mathbb{C}^J \in \Upsilon$, $p(\mathbb{C}^J) = Card\, J$, $q(\mathbb{C}^J) = 0$, *and every exact sequence in* $\mathfrak{M}_E$ *belongs to* $\left(\mathbb{C}^J\right)_\Upsilon$.

a) We put

$$\bar{p} := \sum_{j \otimes J} p(G_j)\,, \qquad \bar{q} := \sum_{j \otimes J} q(G_j)\,.$$

Since

$$F \otimes \prod_{j \in J} G_j \approx \prod_{j \in J} (F \otimes G_j)\,,$$

by Proposition 1.3.3, the maps

$$\prod_{j \in J} \Phi_{i,G_j,F} : K_i(F)^{\bar{p}} \times K_{i+1}(F)^{\bar{q}} = \prod_{j \in J} \Big(K_i(F)^{p(G_j)} \times K_{i+1}(F)^{q(G_j)} \Big) \longrightarrow$$

$$\longrightarrow \prod_{j \in J} K_i(F \otimes G_j) \xrightarrow{\Phi_{(F \otimes G_j)_{j \in J},i}} K_i(F \otimes G)$$

are group isomorphisms. Let $F \overset{\phi}{\to} F'$ be a morphism in $\mathfrak{M}_E$. The diagram

$$\begin{tikzcd}
K_i(F)^{\bar{p}} \times K_{i+1}(F)^{\bar{q}} \arrow[r, "\prod_{j \in J} \Phi_{i,G_j,F}"] \arrow[d, "K_i(\phi)^{\bar{p}} \times K_{i+1}(\phi)^{\bar{q}}"'] & \prod_{j \in J} K_i(F \otimes G_j) \arrow[d, "\prod_{j \in J} K_i\left(\phi \otimes id_{G_j}\right)"] \\
K_i(F')^{\bar{p}} \times K_{i+1}(F')^{\bar{q}} \arrow[r, "\prod_{j \in J} \Phi_{i,G_j,F'}"'] & \prod_{j \in J} K_i(F' \otimes G_j)
\end{tikzcd}$$

is obviously commutative and by Proposition 1.3.4 the diagram

$$\begin{tikzcd}
\prod_{j \in J} K_i(F \otimes G_j) \arrow[r, "\Phi_{(F \otimes G_j)_{j \in J},i}"] \arrow[d, "\prod_{j \in J} K_i\left(\phi \otimes id_{G_j}\right)"'] & K_i(F \otimes G) \arrow[d, "K_i(\phi \otimes id_G)"] \\
\prod_{j \in J} K_i(F' \otimes G_j) \arrow[r, "\Phi_{(F' \otimes G_j)_{j \in J},i}"'] & K_i(F' \otimes G)
\end{tikzcd}$$

is also commutative and this proves the assertion.

b) follows from Proposition 1.5.7 by complete induction.

c) follows from a), b), and Proposition 1.5.4 b). ■

PROPOSITION 1.5.10 *Let J be a finite set and for every $j \in J$ let*

$$0 \longrightarrow F_{j,1} \xrightarrow{\varphi_j} F_{j,2} \xrightarrow{\psi_j} F_{j,3} \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$ and $\delta_{j,i}$ its associated index maps. For every $k \in \{1,2,3\}$ put

$$F_k := \prod_{j\in J} F_{j,k}$$

and for every $j \in J$ denote by

$$\varphi_{j,k} : F_{j,k} \longrightarrow F_k\,, \qquad \psi_{j,k} : F_k \longrightarrow F_{j,k}$$

the canonical inclusion and projection, respectively. Then

$$0 \longrightarrow F_1 \xrightarrow{\prod\limits_{j\in J}\varphi_j} F_2 \xrightarrow{\prod\limits_{j\in J}\psi_j} F_3 \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ and if we denote by δ_i its index maps then the diagram

$$\begin{tikzcd}
\prod\limits_{j\in J} K_i\left(F_{j,3}\right) \arrow[r, "\Psi_{3,i}"] \arrow[d, "\prod\limits_{j\in J}\delta_{j,i}"'] & K_i\left(F_3\right) \arrow[d, "\delta_i"] \\
\prod\limits_{j\in J} K_{i+1}\left(F_{j,1}\right) \arrow[r, "\Psi_{1,(i+1)}"'] & K_{i+1}\left(F_1\right)
\end{tikzcd}$$

is commutative, where for every $k \in \{1,3\}$,

$$\Psi_{k,i} : \prod_{j\in J} K_i\left(F_{j,k}\right) \longrightarrow K_i\left(F_k\right)\,, \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} K_i\left(\varphi_{j,k}\right) a_j\,.$$

For every $j \in J$ the diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{j,1} \arrow[r, "\varphi_j"] \arrow[d, "\varphi_{j,1}"'] & F_{j,2} \arrow[r, "\psi_j"] \arrow[d, "\varphi_{j,2}"'] & F_{j,3} \arrow[r] \arrow[d, "\varphi_{j,3}"] & 0 \\
0 \arrow[r] & F_1 \arrow[r, "\prod\limits_{j\in J}\varphi_j"'] & F_2 \arrow[r, "\prod\limits_{j\in J}\psi_j"'] & F_3 \arrow[r] & 0\,,
\end{tikzcd}$$

is commutative. By the commutativity of the index maps (Axiom 1.2.8), the diagram

$$\begin{tikzcd}
K_i\left(F_{j,3}\right) \arrow[r, "K_i\left(\varphi_{j,3}\right)"] \arrow[d, "\delta_{j,i}"'] & K_i\left(F_3\right) \arrow[d, "\delta_i"] \\
K_{i+1}\left(F_{j,1}\right) \arrow[r, "K_{i+1}\left(\varphi_{j,1}\right)"'] & K_{i+1}\left(F_1\right)
\end{tikzcd}$$

is commutative. Let $(a_j)_{j\in J} \in \prod_{j\in J} K_i\left(F_{j,3}\right)$. Then

$$\delta_i \Psi_{3,i}(a_j)_{j\in J} = \delta_i \sum_{j\in J} K_i\left(\varphi_{j,3}\right) a_j =$$

$$= \sum_{j\in J} K_{i+1}\left(\varphi_{j,1}\right) \delta_{j,i} a_j = \Psi_{1,(i+1)}\left(\prod_{j\in J} \delta_{j,i}\right)(a_j)_{j\in J} .$$

Thus the diagram

$$\begin{array}{ccc}
\prod_{j\in J} K_i\left(F_{j,3}\right) & \xrightarrow{\Psi_{3,i}} & K_i\left(F_3\right) \\
{\scriptstyle \prod_{j\in J}\delta_{j,i}}\big\downarrow & & \big\downarrow{\scriptstyle \delta_i} \\
\prod_{j\in J} K_{i+1}\left(F_{j,1}\right) & \xrightarrow[\Psi_{1,(i+1)}]{} & K_{i+1}\left(F_1\right)
\end{array}$$

is commutative. ■

PROPOSITION 1.5.11 *Let $(G_j)_{j\in J}$ be a finite family in Υ.*

a)

$$G := \bigotimes_{j\in J} G_j \in \Upsilon ,$$

$$p(G) = \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j)) + \prod_{j\in J}(p(G_j)-q(G_j))\right) ,$$

$$q(G) = \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j)) - \prod_{j\in J}(p(G_j)-q(G_j))\right) .$$

b) If G_{j_0} is K-null for a $j_0 \in J$ then $F \otimes \left(\bigotimes_{j\in J} G_j\right)$ is also K-null.

c) If $p(G_{j_0}) = q(Gj_0)$ for a $j_0 \in J$ then $p(G) = q(G)$.

d) Let $j_0 \in J$, $J' := J \setminus \{j_0\}$, and $G' := \bigotimes_{j\in J'} G_j$.

d_1) If $p(G_{j_0}) = 1$, $q(G_{j_0}) = 0$ then $p(G') = p(G)$, $q(G') = q(G)$.

d_2) If $p(G_{j_0}) = 0$, $q(G_{j_0}) = 1$ then $p(G') = q(G)$, $q(G') = p(G)$.

e) If we put

$$H := \bigotimes_{j\in J} \tilde{G}_j \qquad \text{and} \qquad G_I := \bigotimes_{j\in I} G_j$$

for every $I \subset J$ then

$$H \in \Upsilon, \qquad p(H) = \sum_{I\subset J} p(G_I), \qquad q(H) = \sum_{I\subset J} q(G_I);.$$

f) If in addition G and all $(G_j)_{j\in J}$ are nuclear then

$$\bigcap_{j\in J} (G_j)_\Upsilon \subset G_\Upsilon .$$

a) Assume first $J = \{1,2\}$. The maps

$$K_i(F)^{p(G_1)p(G_2)+q(G_1)q(G_2)} \times K_{i+1}(F)^{p(G_1)q(G_2)+p(G_2)q(G_1)} =$$

$$= \left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)}\right)^{p(G_2)} \times \left(K_{i+1}(F)^{p(G_1)} \times K_i(F)^{q(G_1)}\right)^{q(G_2)} \longrightarrow$$

$$\xrightarrow{(\Phi_{i,G_1,F})^{p(G_2)} \times (\Phi_{(i+1),G_1,F})^{q(G_2)}}$$

$$\longrightarrow K_i(F \otimes G_1)^{p(G_2)} \times K_{i+1}(F \otimes G_1)^{q(G_2)} \longrightarrow$$

$$\xrightarrow{\Phi_{i,G_2,F\otimes G_1}}$$

$$\longrightarrow K_i((F \otimes G_1) \otimes G_2) \approx K_i(F \otimes (G_1 \otimes G_2))$$

are group isomorphisms and

$$p(G_1 \otimes G_2) := p(G_1)p(G_2) + q(G_1)q(G_2) =$$

$$= \frac{1}{2}[(p(G_1)+q(G_1))(p(G_2)+q(G_2)) + (p(G_1)-q(G_1))(p(G_2)-q(G_2))],$$

$$q(G_1 \otimes G_2) := p(G_1)q(G_2) + p(G_2)q(G_1) =$$

$$= \frac{1}{2}[(p(G_1)+q(G_1))(p(G_2)+q(G_2)) - (p(G_1)-q(G_1))(p(G_2)-q(G_2))].$$

If $F \xrightarrow{\phi} F'$ is a morphism in $\mathfrak{M}_E$ then the diagrams

$$\begin{array}{ccc}
K_i(F \otimes G_1)^{p(G_2)} \times K_{i+1}(F \otimes G_1)^{q(G_2)} & \xrightarrow{\Phi_{i,G_2,(F\otimes G_1)}} & K_i((F \otimes G_1) \otimes G_2) \\
\Big\downarrow {\scriptstyle K_i\left(\phi\otimes id_{G_1}\right)^{p(G_2)} \times K_{i+1}\left(\varphi\otimes id_{G_1}\right)^{q(G_2)}} & & \Big\downarrow {\scriptstyle K_i\left((\phi\otimes id_{G_1})\otimes id_{G_2}\right)} \\
K_i(F' \otimes G_1)^{p(G_2)} \times K_{i+1}(F' \otimes G_1)^{q(G_2)} & \xrightarrow[\Phi_{i,G_2,(F'\otimes G_1)}]{} & K_i((F' \otimes G_1) \otimes G_2)
\end{array}$$

$$\begin{array}{ccc} K_i\left((F\otimes G_1)\otimes G_2\right) & \xrightarrow{\approx} & K_i\left(F\otimes(G_1\otimes G_2)\right) \\ \Big\downarrow {\scriptstyle K_i\left((\phi\otimes id_{G_1})\otimes id_{G_2}\right)} & & \Big\downarrow {\scriptstyle K_i\left(\phi\otimes id_{(G_1\otimes G_2)}\right)} \\ K_i\left((F'\otimes G_1)\otimes G_2\right) & \xrightarrow[\approx]{} & K_i\left(F'\otimes(G_1\otimes G_2)\right) \end{array}$$

are commutative, which proves the assertion in this case.

The general case is obtained now by induction with respect to $Card\,J$. Let $Card\,J > 1$, $k \in J$, $J' := J \setminus \{\text{k}\}$, $G' := \otimes_{j\in J'} G_j$ and assume the assertion holds for J'. By the above,

$$p(G) = \frac{1}{2}[(p(G')+q(G'))(p(G_k)+q(G_k))+(p(G')-q(G'))(p(G_k)-q(G_k))] =$$

$$= \frac{1}{2}\left(\prod_{j\in J'}(p(G_j)+q(G_j))(p(G_k)+q(G_k))+\right.$$

$$\left.+\prod_{j\in J'}(p(G_j)-q(G_j))(p(G_k)-q(G_k))\right) =$$

$$= \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j))+\prod_{j\in J}(p(G_j)-q(G_j))\right),$$

$$q(G) = \frac{1}{2}[(p(G')+q(G'))(p(G_k)+q(G_k))-(p(G')-q(G'))(p(G_k)-q(G_k))] =$$

$$= \frac{1}{2}\left(\prod_{j\in J'}(p(G_j)+q(G_j))(p(G_k)+q(G_k))-\right.$$

$$\left.-\prod_{j\in J'}(p(G_j)-q(G_j))(p(G_k)-q(G_k))\right) =$$

$$= \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j))-\prod_{j\in J}(p(G_j)-q(G_j))\right).$$

b), c), and d) follow directly from a).

e) By Corollary 1.5.8, $\tilde{G}_j \in \Upsilon$ for every $j \in J$. By a) and Proposition 1.4.6, $H \in \Upsilon$,

$$K_i(F\otimes H) \approx \prod_{I\subset J} K_i(F\otimes G_I) \approx \prod_{I\subset J}\left(K_i(F)^{p(G_I)} \times K_{i+1}(F)^{q(G_I)}\right) =$$

$$= K_i(F)^{\sum\limits_{I\subset J} p(G_I)} \times K_{i+1}(F)^{\sum\limits_{I\subset J} q(G_I)}.$$

f) Assume first $J := \{1,2\}$, let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in (G_1)_\Upsilon \cap (G_2)_\Upsilon ,$$

and let δ_i be its index maps. Then (by a)) the diagram

$$\begin{tikzcd}
K_i\left(F_3\right)^{p(G)} \times K_{i+1}\left(F_3\right)^{q(G)} \arrow[r, "\delta_i^{p(G)} \times \delta_{i+1}^{q(G)}"] \arrow[d, "\Phi^{p(G_2)}_{i,G_1,F_3} \times \Phi^{q(G_2)}_{(i+1),G_1,F_3}"'] & A \arrow[d, "\Phi^{p(G_2)}_{(i+1),G_1,F_1} \times \Phi^{q(G_2)}_{i,G_1,F_1}"'] \\
K_i\left(F_3 \otimes G_1\right)^{p(G_2)} \times K_{i+1}\left(F_3 \otimes G_1\right)^{q(G_2)} \arrow[r, "\delta^{p(G_1)}_{G_1,i} \times \delta^{q(G_2)}_{G_1,i}"] \arrow[d, "\Phi_{i,G_2,(F_3 \otimes G_1)}"'] & B \arrow[d, "\Phi_{(i+1),G_2,(F_1 \otimes G_1)}"'] \\
K_i\left(\left(F_3 \otimes G_1\right) \otimes G_2\right) \approx K_i\left(F_3 \otimes \left(G_1 \otimes G_2\right)\right) \arrow[r, "\delta_{G_1,(G_2,i)}"'] & C
\end{tikzcd}$$

is commutative, where

$$A := K_{i+1}\left(F_1\right)^{p(G)} \times K_i\left(F_1\right)^{q(G)} ,$$

$$B := K_{i+1}\left(F_1 \otimes G_1\right)^{p(G_2)} \times K_i\left(F_1 \otimes G_1\right)^{q(G_2)} ,$$

$$C := K_{i+1}\left(\left(\left(F_1 \otimes G_1\right) \otimes G_2\right)\right) \approx K_{i+1}\left(\left(F_1 \otimes \left(G_1 \otimes G_2\right)\right)\right) .$$

Thus

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in G_\Upsilon .$$

The general case follows by induction with respect to $Card\, J$. ■

COROLLARY 1.5.12 *Let $G \in \Upsilon$, $n \in \mathbb{N}$, and $H := \otimes_{j \in \mathbb{N}_n} G$. Then $H \in \Upsilon$, $G_\Upsilon \subset H_\Upsilon$, and*

$$p(H) = \frac{1}{2}\left((p(G)+q(G))^n + (p(G)-q(G))^n\right) ,$$

$$q(H) = \frac{1}{2}\left((p(G)+q(G))^n - (p(G)-q(G))^n\right) .$$

The assertion follows from Proposition 1.5.11 a). ■

PROPOSITION 1.5.13 *Let (G_1, G_2, G_3) be an $\mathfrak{M}_{\mathbf{C}}$-triple such that G_1/G_3 and G_2/G_3 are nuclear, G_2 is Υ-null, and $G_1, G_3 \in \Upsilon$. We use the notation of the triple theorem* (Theorem 1.3.8 a)) *associated to the $\mathfrak{M}_E$-triple*

$$(F \otimes G_1, F \otimes G_2, F \otimes G_3)$$

(Proposition 1.4.9), *put* $\varphi := \varphi_{1,2}/(F\otimes G_3)$ (as in Proposition 1.3.7 a)), *and denote by*

$$\Psi_{F,i} : K_i\left(F\otimes G_1\right)\times K_{i+1}\left(F\otimes G_3\right)\longrightarrow K_i\left(F\otimes\left(G_1/G_3\right)\right),$$

$$(a,b)\longmapsto K_i\left(\psi_{1,3}\right)a+\Phi_i b$$

the corresponding group isomorphism (Theorem 1.3.8 a_4), Proposition 1.4.9). *Then*

$$G_1/G_3\in\Upsilon,\qquad p(G_1/G_3)=p(G_1)+q(G_3),\qquad q(G_1/G_3)=q(G_1)+p(G_3),$$

$$\Phi_{i,(G_1/G_3),F}=\Psi_{F,i}\circ\left(\Phi_{i,G_1,F}\times\Phi_{(i+1),G_3,F}\right).$$

Since $G_1,G_3\in\Upsilon$, the map

$$\Psi_{F,i}\circ\left(\Phi_{i,G_1,F}\times\Phi_{(i+1),G_3,F}\right):\left(K_i\left(F\right)^{p(G_1)}\times K_{i+1}\left(F\right)^{q(G_1)}\right)\times$$

$$\times\left(K_{i+1}\left(F\right)^{p(G_3)}\times K_i\left(F\right)^{q(G_3)}\right)\longrightarrow K_i\left(F\otimes\left(G_1/G_3\right)\right)$$

is a group isomorphism. We put

$$\bar{p}(G_1/G_3):=p(G_1)+Q(G_3),\qquad\qquad \bar{q}(G_1/G_3):=q(G_1)+p(G_3),$$

$$\bar{\Phi}_{i,G_1/G_3,F}:=\Psi_{F,i}\circ\left(\Phi_{i,G_1,F}\times\Phi_{(i+1),G_3,F}\right).$$

Let $F\xrightarrow{\phi}F'$ be a morphism in $\mathfrak{M}_E$. We mark with a prime the above notation associated to F'. By the commutativity of the index maps (Axiom 1.2.8),

$$K_{i+1}\left(\phi\otimes id_{G_3}\right)\circ\delta_{2,3,i}=\delta'_{2,3,i}\circ K_i\left(\phi\otimes id_{(G_2/G_3)}\right).$$

Moreover

$$K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\circ K_i\left(\varphi\right)=K_i\left(\varphi'\right)\circ K_i\left(\phi\otimes id_{(G_2/G_3)}\right),$$

$$K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\circ K_i\left(\psi_{1,3}\right)=K_i\left(\psi'_{1,3}\right)\circ K_i\left(\phi\otimes id_{G_1}\right).$$

It follows

$$K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\circ\Phi_i=K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\circ K_i\left(\varphi\right)\circ\left(\delta_{2,3,i}\right)^{-1}=$$

$$=K_i\left(\varphi'\right)\circ K_i\left(\phi\otimes id_{(G_2/G_3)}\right)\circ\left(\delta_{2,3,i}\right)^{-1}=$$

$$=K_i\left(\varphi'\right)\circ\left(\delta'_{2,3,i}\right)^{-1}\circ K_{i+1}\left(\phi\otimes id_{G_3}\right)=\Phi'_i\circ K_{i+1}\left(\phi\otimes id_{G_3}\right).$$

We want to prove that the diagram

$$\begin{array}{ccc} K_i(F\otimes G_1)\times K_{i+1}(F\otimes G_3) & \xrightarrow{\Psi_{F,i}} & K_i(F\otimes(G_1/G_3)) \\ {\scriptstyle K_i(\phi\otimes id_{G_1})\times K_{i+1}(\phi\otimes id_{G_3})}\Big\downarrow & & \Big\downarrow{\scriptstyle K_i\left(\phi\otimes id_{(G_1/G_3)}\right)} \\ K_i(F'\otimes G_1)\times K_{i+1}(F'\otimes G_3) & \xrightarrow[\Psi_{F',i}]{} & K_i(F'\otimes(G_1/G_3)) \end{array}$$

is commutative. For $(a,b)\in K_i(F\otimes G_1)\times K_{i+1}(F\otimes G_3)$, by the above,

$$K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\Psi_{F,i}(a,b)=K_i\left(\phi\otimes id_{(G_1/G_3)}\right)(K_i(\psi_{1,3})a+\Phi_i b)=$$

$$=K_i\left(\phi\otimes id_{(G_1/G_3)}\right)K_i(\psi_{1,3})a+K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\Phi_i b=$$

$$=K_i\left(\psi'_{1,3}\right)K_i(\phi\otimes id_{G_1})a+\Phi'_i K_{i+1}\left(\phi\otimes id_{G_3}\right)b=$$

$$=\Psi_{F',i}(K_i(\phi\otimes id_{G_1})a,K_{i+1}\left(\phi\otimes id_{G_3}\right)b)=$$

$$=\Psi_{F',i}(K_i(\phi\otimes id_{G_1})\times K_{i+1}\left(\phi\otimes id_{G_3}\right))(a,b)\ .$$

Thus the above diagram is commutative. It follows, since $G_1,G_3\in\Upsilon$, that the diagram

$$\begin{array}{ccc} K_i(F)^{p(G_1/G_3)}\times K_{i+1}(F)^{q(G_1/G_3)} & \xrightarrow{\bar{\Phi}_{i,(G_1/G_3),F}} & K_i(F\otimes(G_1/G_3)) \\ {\scriptstyle K_i(\phi)^{p(G_1/G_3)}\times K_{i+1}(\phi)^{q(G_1/G_3)}}\Big\downarrow & & \Big\downarrow{\scriptstyle K_i\left(\phi\otimes id_{(G_1/G_3)}\right)} \\ K_i(F')^{p(G_1/G_3)}\times K_{i+1}(F')^{q(G_1/G_3)} & \xrightarrow[\bar{\Phi}_{i,(G_1/G_3),F'}]{} & K_i(F'\otimes(G_1/G_3)) \end{array}$$

is commutative. Hence

$$G_1/G_3\in\Upsilon,\qquad p(G_1/G_3)=p(G_1)+q(G_3),\qquad q(G_1/G_3)=q(G_1)+p(G_3),$$

$$\Phi_{i,(G_1/G_3),F}=\Psi_{F,i}\circ(\Phi_{i,G_1,F}\times\Phi_{(i+1),G_3,F})\ .$$ ■

PROPOSITION 1.5.14 *Let* (G_1,G_2,G_3) *be an* $\mathfrak{M}_{\mathbf{C}}$*-triple such that* G_1/G_2 *and* G_1/G_3 *are nuclear,* G_1/G_3 *is* Υ*-null, and* $G_1,G_1/G_2\in\Upsilon$*. We use the notation of the triple theorem* (Theorem 1.3.8 b)) *associated to the* $\mathfrak{M}_E$*-triple*

$$(F\otimes G_1,F\otimes G_2,F\otimes G_3)$$

(Proposition 1.4.9)*, assume* ψ_{12} *K-null for all E-C*-algebras F, and denote by*

$$\Psi_{F,i}:K_i(F\otimes G_1)\times K_{i+1}(F\otimes(G_1/G_2))\longrightarrow K_i(F\otimes G_2),$$

$$(a,b) \longmapsto \Phi_i' a + \delta_{1,2,(i+1)} b$$

the corresponding group isomorphism (Theorem 1.3.8 b_4), Proposition 1.4.9). *Then*

$$G_2 \in \Upsilon, \qquad p(G_2) = p(G_1) + q(G_1/G_2), \qquad q(G_2) = q(G_1) + p(G_1/G_2),$$

$$\Phi_{i,G_2,F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) \, .$$

Since $G_1, G_1/G_2 \in \Upsilon$, the map

$$\Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) : \left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)} \right) \times$$
$$\times \left(K_{i+1}(F)^{p(G_1/G_2)} \times K_i(F)^{q(G_1/G_2)} \right) \longrightarrow K_i(F \otimes G_2)$$

is a group isomorphism. We put

$$\tilde{p}(G_2) := p(G_1) + q(G_1/G_2), \qquad \tilde{q}(G_2) := q(G_1) + p(G_1/G_2),$$

$$\tilde{\Phi}_{i,G_2,F} := \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) \, .$$

Let $F \xrightarrow{\phi} \bar{F}$ be a morphism in $\mathfrak{M}_E$. We mark with a bar the above notation associated to $\bar{F}$. By the commutativity of the index maps (Axiom 1.2.8),

$$K_i(\phi \otimes id_{G_2}) \circ \delta_{1,2,(i+1)} = \bar{\delta}_{1,2,(i+1)} \circ K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right) \, .$$

Moreover

$$K_i(\phi \otimes id_{G_1}) \circ K_i(\varphi_{1,3}) = K_i(\bar{\varphi}_{1,3}) \circ K_i\left(\phi \otimes id_{G_3}\right) ,$$
$$K_i(\phi \otimes id_{G_2}) \circ K_i(\varphi_{2,3}) = K_i(\bar{\varphi}_{2,3}) \circ K_i\left(\phi \otimes id_{G_3}\right) .$$

It follows

$$K_i(\phi \otimes id_{G_2}) \circ \Phi_i' = K_i(\phi \otimes id_{G_2}) \circ K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1} =$$
$$= K_i(\bar{\varphi}_{2,3}) \circ K_i\left(\phi \otimes id_{G_3}\right) \circ K_i(\varphi_{1,3})^{-1} =$$
$$= K_i(\bar{\varphi}_{2,3}) \circ K_i(\bar{\varphi}_{1,3})^{-1} \circ K_i(\phi \otimes id_{G_1}) = \bar{\Phi}_i' \circ K_i(\phi \otimes id_{G_1}) \, .$$

We want to prove that the diagram

$$\begin{array}{ccc}
K_i(F \otimes G_1) \times K_{i+1}(F \otimes (G_1/G_2)) & \xrightarrow{\Psi_{F,i}} & K_i(F \otimes G_2) \\
{\scriptstyle K_i(\phi \otimes id_{G_1}) \times K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right)} \Big\downarrow & & \Big\downarrow {\scriptstyle K_i(\phi \otimes id_{G_2})} \\
K_i(\bar{F} \otimes G_1) \times K_{i+1}(\bar{F} \otimes (G_1/G_2)) & \xrightarrow[\Psi_{\bar{F},i}]{} & K_i(\bar{F} \otimes G_2)
\end{array}$$

is commutative. For $(a,b) \in K_i(F \otimes G_1) \times K_{i+1}(F \otimes (G_1/G_2))$, by the above,

$$K_i(\phi \otimes id_{G_2})\Psi_{F,i}(a,b) = K_i(\phi \otimes id_{G_2})(\Phi'_i a + \delta_{1,2,(i+1)} b) =$$

$$= K_i(\phi \otimes id_{G_2})\Phi'_i a + K_i(\phi \otimes id_{G_2})\delta_{1,2,(i+1)} b =$$

$$= \bar{\Phi}'_i K_i(\phi \otimes id_{G_1}) a + \bar{\delta}_{1,2,(i+1)} K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right) b =$$

$$= \Psi_{\bar{F},i}(K_i(\phi \otimes id_{G_1}) a, K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right) b) =$$

$$= \Psi_{\bar{F},i}\left(K_i(\phi \otimes id_{G_1}) \times K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right)\right)(a,b) .$$

Thus the above diagram is commutative. Since $G_1, G_1/G_2 \in \Upsilon$, It follows that the diagram

$$\begin{array}{ccc}
K_i(F)^{\tilde{p}(G_2)} \times K_{i+1}(F)^{\tilde{q}(G_2)} & \xrightarrow{\tilde{\Phi}_{i,G_2,F}} & K_i(F \otimes G_2) \\
{\scriptstyle K_i(\phi)^{p(G_2)} \times K_{i+1}(\phi)^{q(G_2)}} \downarrow & & \downarrow {\scriptstyle K_i\left(\phi \otimes id_{G_2}\right)} \\
K_i(\bar{F})^{\tilde{p}(G_2)} \times K_{i+1}(\bar{F})^{\tilde{q}(G_2)} & \xrightarrow[\tilde{\Phi}_{i,G_2,\bar{F}}]{} & K_i(\bar{F} \otimes G_2)
\end{array}$$

is commutative. Hence

$$G_2 \in \Upsilon, \qquad p(G_2) = p(G_1) + q(G_1/G_2), \qquad q(G_2) = q(G_1) + p(G_1/G_2),$$

$$\Phi_{i,G_2,F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) . \qquad \blacksquare$$

PROPOSITION 1.5.15 *Let*

$$0 \longrightarrow G \xrightarrow{\varphi} H \xrightarrow{\psi} \mathbb{C} \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbb{C}}$ *with* G *nuclear and* H Υ*-null and let* δ_i^F *denote the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0 .$$

Then

$$G \in \Upsilon, \qquad p(G) = 0, \qquad q(G) = 1, \qquad \Phi_{i,G,F=\delta_{i+1}^F} ,$$

$$(0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0) \in G_{\Upsilon} .$$

By Proposition 1.5.6 b) and Proposition 1.5.4 b),

$$G \in \Upsilon, \qquad p(G) = 0, \qquad q(G) = 1, \qquad \Phi_{i,G,F=\delta^F_{i+1}} .$$

Since the diagram

$$\begin{array}{ccc} K_{i+1}(F) & \xrightarrow{\delta^F_{i+1}} & K_i(F \otimes G) \\ {\scriptstyle \Phi_{i,G,F}=\delta^F_{i+1}} \big\downarrow & & \big\downarrow {\scriptstyle \Phi_{(i+1),G,(F\otimes G)}=\delta^F_{G,i}} \\ K_i(F \otimes G) & \xrightarrow[\delta^F_{G,i}]{} & K_{i+1}((F \otimes G) \otimes G) \end{array}$$

is obviously commutative,

$$(0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0) \in G_\Upsilon .$$

■

1.6 The Class Υ_1

Throughout this section F denotes an E-C*-algebra.

DEFINITION 1.6.1 *We denote by Υ_1 the class of unital C*-algebras G belonging to Υ such that*

$$p(G) = 1\,, \qquad q(G) = 0\,, \qquad \Phi_{i,G,F} = K_i\,(\phi_{G,F})\,,$$

where

$$\phi_{G,F} : F \longrightarrow F \otimes G\,, \quad x \longmapsto x \otimes 1_G\,.$$

PROPOSITION 1.6.2 $\mathbb{C} \in \Upsilon_1$.

In fact

$$\phi_{\mathbb{C},F} : F \longrightarrow F \otimes \mathbb{C}\,, \quad x \longmapsto x \otimes 1_{\mathbb{C}}$$

is an isomorphism. ■

PROPOSITION 1.6.3 *Let $G \in \Upsilon_1$ and let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. If we identify $K_i\,(F)$ with $K_i\,(F \otimes G)$ for all E-C*-algebras F using the group isomorphisms $\Phi_{i,G,F}$ then $K_i\,(\phi \otimes id_G)$ is identified with $K_i\,(\phi)$.*

The assertion follows from the commutativity of the diagram

$$\begin{array}{ccc} K_i(F) & \xrightarrow{K_i(\phi)} & K_i(F') \\ \Phi_{i,G,F}\big\downarrow & & \big\downarrow\Phi_{i,G,F'} \\ K_i(F\otimes G) & \xrightarrow[K_i(\phi\otimes id_G)]{} & K_i(F'\otimes G) \end{array}$$ ■

PROPOSITION 1.6.4 *Let G,H be C*-algebras and $\varphi : G \longrightarrow H$ and $\psi : H \longrightarrow G$ be a homotopy such that φ and ψ are unital. If $G \in \Upsilon_1$ then $H \in \Upsilon_1$.*

By Proposition 1.5.4 c),

$$H \in \Upsilon, \qquad p(H) = 1, \qquad q(H) = 0,$$

$$\Phi_{i,H,F} = K_i\left(id_F \otimes \varphi\right) \circ \Phi_{i,G,F} = K_i\left(id_F \otimes \varphi\right) \circ K_i\left(\phi_{G,F}\right) = K_i\left(\phi_{H,F}\right) .$$ ■

PROPOSITION 1.6.5 *If $(G_j)_{j\in J}$ is a finite family in Υ_1, $J \neq \emptyset$, then $\bigotimes_{j\in J} G_j \in \Upsilon_1$.*

$\bigotimes_{j\in J} G_j$ is unital and by Proposition 1.5.12 a), $\bigotimes_{j\in J} G_j \in \Upsilon$. Assume $J = \{1,2\}$ and let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. Then the diagram

$$\begin{array}{ccccc} K_i\left(F\right) & \xrightarrow{K_i\left(\phi_{G_1,F}\right)} & K_i\left(F\otimes G_1\right) & \xrightarrow{K_i\left(\phi_{G_2,(F\otimes G_1)}\right)} & K_i\left(F\otimes G_1\otimes G_2\right) \\ K_i(\phi)\big\downarrow & & \big\downarrow K_i\left(\phi\otimes id_{G_1}\right) & & \big\downarrow K_i\left(\phi\otimes id_{(G_1\otimes G_2)}\right) \\ K_i\left(F'\right) & \xrightarrow[K_i\left(\phi_{G_1,F'}\right)]{} & K_i\left(F'\otimes G_1\right) & \xrightarrow[K_i\left(\phi_{G_2,(F'\otimes G_1)}\right)]{} & K_i\left(F'\otimes G_1\otimes G_2\right) \end{array}$$

is commutative. Since

$$\phi_{(G_1\otimes G_2),F} = \phi_{G_2,(F\otimes G_1)} \circ \phi_{G_1,F} \,, \qquad \phi_{(G_1\otimes G_2),F'} = \phi_{G_2,(F'\otimes G_1)} \circ \phi_{G_1,F'} \,,$$

the diagram

$$\begin{array}{ccc} K_i\left(F\right) & \xrightarrow{K_i\left(\phi_{(G_1\otimes G_2),F}\right)} & K_i\left(F\otimes G_1\otimes G_2\right) \\ K_i(\phi)\big\downarrow & & \big\downarrow K_i\left(\phi\otimes id_{(G_1\otimes G_2)}\right) \\ K_i\left(F'\right) & \xrightarrow[K_i\left(\phi_{(G_1\otimes G_2),F'}\right)]{} & K_i\left(F'\otimes G_1\otimes G_2\right) \end{array}$$

is commutative, which proves the assertion in this case. The general case follows now by induction with respect to $Card\,J$. ■

PROPOSITION 1.6.6 *If $G \in \Upsilon_1$ is nuclear then every exact sequence in $\mathfrak{M}_E$ belongs to G_Υ.*

Let

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$. Then the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & F_2 & \xrightarrow{\phi_2} & F_3 & \longrightarrow & 0 \\
 & & {\scriptstyle \phi_{G,F_1}}\big\downarrow & & {\scriptstyle \phi_{G,F_2}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{G,F_3}} & & \\
0 & \longrightarrow & F_1 \otimes G & \xrightarrow[\phi_1 \otimes id_G]{} & F_2 \otimes G & \xrightarrow[\phi_2 \otimes id_G]{} & F_3 \otimes G & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8) the diagram

$$\begin{array}{ccc}
K_i(F_3) & \xrightarrow{\delta_i} & K_{i+1}(F_1) \\
{\scriptstyle \Phi_{i,G,F_3}=K_i(\phi_{G,F_3})}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{(i+1),G,F_1}=K_{i+1}(\phi_{G,F_1})} \\
K_i(F_3 \otimes G) & \xrightarrow[\delta_{G,i}]{} & K_{i+1}(F_1 \otimes G)
\end{array}$$

is commutative, where δ_i denotes the index maps of the exact sequence

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0\,.$$ ■

PROPOSITION 1.6.7 *Let G be a C*-algebra.*

a) $\phi_{\tilde{G},F} = (id_F \otimes \lambda_G) \circ \phi_{\mathbf{C},F}$.

b) G is Υ-null iff $\tilde{G} \in \Upsilon_1$.

c) If G is Υ-null and $\varphi : G \longrightarrow G'$, $\psi : G \longrightarrow G'$ are C-homomorphisms then $K_i\,(id_F \otimes \tilde{\varphi}) = K_i\,(id_F \otimes \tilde{\psi})$. In particular if $G = G'$ then*

$$K_i\,(id_F \otimes \tilde{\varphi}) = id_{K_i\left(F \otimes \tilde{G}\right)} \approx id_{K_i(F)}\,.$$

a) is easy to see.

b) By Corollary 1.4.5 b), the sequence

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(id_F \otimes \iota_G)} K_i\left(F \otimes \tilde{G}\right) \underset{\xleftarrow{K_i(id_F \otimes \lambda_G)}}{\xrightarrow{K_i(id_F \otimes \pi_G)}} K_i(F) \longrightarrow 0$$

is split exact. By a) and Proposition 1.6.2,

$$K_i\left(\phi_{\tilde{G},F}\right) = K_i\left(id_F \otimes \lambda_G\right) \circ \Phi_{i,\mathbf{C},F} \ .$$

If $\tilde{G} \in \Upsilon_1$ then

$$\Phi_{i,\tilde{G},F} = K_i\left(\phi_{\tilde{G},F}\right) = K_i\left(id_F \otimes \lambda_G\right) \circ \Phi_{i,\mathbf{C},F} \ ,$$

so by Proposition 1.6.2, $K_i\left(id_F \otimes \lambda_G\right)$ is an isomorphism, $K_i\left(id_F \otimes \iota_G\right) = 0$, $K_i\left(F \otimes G\right) = 0$, and G is Υ-null. If G is Υ-null then $K_i\left(id_F \otimes \lambda_G\right)$ is an isomorphism so

$$K_i\left(\phi_{\tilde{G},F}\right) : K_i\left(F\right) \longrightarrow K_i\left(F \otimes G\right)$$

is an isomorphism and $\tilde{G} \in \Upsilon_1$.

c) Since $\tilde{\varphi} \circ \lambda_G = \tilde{\psi} \circ \lambda_G$,

$$K_i\left(id_F \otimes \tilde{\varphi}\right) \circ K_i\left(id_F \otimes \lambda_G\right) = K_i\left(id_F \otimes \tilde{\psi}\right) \circ K_i\left(id_F \otimes \lambda_G\right) \ .$$

By b), $\tilde{G} \in \Upsilon_1$ and so $K_i\left(id_F \otimes \lambda_G\right)$ is an isomorphism. Thus $K_i\left(id_F \otimes \tilde{\varphi}\right) = K_i\left(id_F \otimes \tilde{\psi}\right)$. ∎

COROLLARY 1.6.8 *If $(G_j)_{j \in J}$ is a finite family of Υ-null C*-algebras and $G := \prod\limits_{j \in J} G_j$ then $\tilde{G} \in \Upsilon_1$.*

By Proposition 1.5.9 a), G is Υ-null and by Proposition 1.6.7 b), $\tilde{G} \in \Upsilon_1$. ∎

PROPOSITION 1.6.9 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_\mathbf{C}$ such that G_1 is Υ-null, G_3 is nuclear, and G_2, G_3 are unital. Then $G_2 \in \Upsilon_1$ iff $G_3 \in \Upsilon_1$.

Since G_2 and G_3 are unital and ψ is surjective, $\psi(1_{G_2}) = 1_{G_3}$. It follows

$$\phi_{G_3,F} = (id_F \otimes \psi) \circ \phi_{G_2,F}\,, \qquad K_i\left(\phi_{G_3,F}\right) = K_i\left(id_F \otimes \psi\right) \circ K_i\left(\phi_{G_2,F}\right)\,.$$

By Proposition 1.5.6 a), $K_i\left(id_F \otimes \psi\right)$ is a group isomorphism,

$$G_2, G_3 \in \Upsilon\,, \qquad p(G_2) = p(G_3) = 1\,, \qquad q(G_2) = q(G_3) = 0\,,$$

$$\Phi_{i,G_3,F} = K_i\left(id_F \otimes \psi\right) \circ \Phi_{i,G_2,F}\,.$$

If $G_2 \in \Upsilon_1$ then by the above,

$$\Phi_{i,G_3,F} = K_i\left(id_F \otimes \psi\right) \circ K_i\left(\phi_{G_2,F}\right) = K_i\left(\phi_{G_3,F}\right)\,,$$

so $G_3 \in \Upsilon_1$. If $G_3 \in \Upsilon_1$ then by the above,

$$K_i\left(id_F \otimes \psi\right) \circ K_i\left(\phi_{G_2,F}\right) = K_i\left(\phi_{G_3,F}\right) = \Phi_{i,G_3,F} = K_i\left(id_F \otimes \psi\right) \circ \Phi_{i,G_2,F}\,,$$

so $\Phi_{i,G_2,F} = K_i\left(\phi_{G_2,F}\right)$ and $G_2 \in \Upsilon_1$. ∎

Chapter 2

Locally Compact Spaces

2.1 Tietze's Theorem

DEFINITION 2.1.1 *Let Ω be a topological space and F an E-C*-algebra. We endow canonically the C*-algebra $\mathscr{C}(\Omega, F)$ with the structure of an E-C*-algebra by putting*

$$\alpha x : \Omega \longrightarrow F\,, \quad \omega \longmapsto \alpha x(\omega)$$

for all $(\alpha, x) \in E \times F$. If Ω is a locally compact space then we endow $\mathscr{C}_0(\Omega, F)$ with the structure of on $E-C*-algebra$ in a similar way. If Ω' is an open set of a locally compact space Ω then we identify $\mathscr{C}_0(\Omega', F)$ with the E-ideal $\{\, x \in \mathscr{C}_0(\Omega, F) \mid x|(\Omega \setminus \Omega') = 0 \,\}$ of $\mathscr{C}_0(\Omega, F)$.

DEFINITION 2.1.2 *Let Ω be a locally compact space with $\mathscr{C}_0(\Omega, \mathbb{C}) \in \Upsilon$. We put*

$$\Omega \in \Upsilon\,, \qquad p(\Omega) := p(\mathscr{C}_0(\Omega, \mathbb{C}))\,, \qquad q(\Omega) := q(\mathscr{C}_0(\Omega, \mathbb{C}))\,,$$

$$\Phi_{i,\Omega,F} := \Phi_{i,\mathscr{C}_0(\Omega,\mathbb{C}),F}\,, \quad \Omega_\Upsilon := \mathscr{C}_0(\Omega, \mathbb{C})_\Upsilon\,, \quad \Omega \in \Upsilon_1 :\Longleftrightarrow \mathscr{C}_0(\Omega, \mathbb{C}) \in \Upsilon_1\,.$$

We say that Ω is **Υ-null** *if $\mathscr{C}_0(\Omega, \mathbb{C})$ is Υ-null. We say that Ω is* **null-homotopic** *if $\mathscr{C}_0(\Omega, \mathbb{C})$ is null-homotopic.*

PROPOSITION 2.1.3 *If Ω is a locally compact space and if Ω^* denotes its Alexandroff compactification then Ω is Υ-null iff $\Omega^* \in \Upsilon_1$.*

The Proposition is a particular case of Proposition 1.6.7. ■

LEMMA 2.1.4 *Let Ω be a locally compact space.*

a) $\mathscr{C}_0(\Omega, \mathbb{C})$ *is nuclear.*

b) $\mathscr{C}_0(\Omega, F) \approx F \otimes \mathscr{C}_0(\Omega, \mathbb{C})$.

c) If Ω is a finite compact space then $\Omega \in \Upsilon$, $p(\Omega) = Card\,\Omega$, $q(\Omega) = 0$, and every exact sequence in $\mathfrak{M}_E$ belongs to Ω_Υ.

a) [5] Theorem T.6.20.

b) [5] Proposition T.5.11,

c) follows from Proposition 1.5.9 c). ■

COROLLARY 2.1.5 (Tietze's Theorem) *Let Ω be a locally compact space, Γ a closed set of Ω, $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ the inclusion map, and*

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Gamma, F), \quad x \longmapsto x|\Gamma .$$

Then

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0(\Gamma, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$.

By Lemma 2.1.4 a),b), the assertion follows from Proposition 1.4.9. ■

COROLLARY 2.1.6 *If*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ and Ω a locally compact space then

$$0 \longrightarrow \mathscr{C}_0(\Omega, F_1) \xrightarrow{\phi_1 \otimes id_G} \mathscr{C}_0(\Omega, F_2) \xrightarrow{\phi_2 \otimes id_G} \mathscr{C}_0(\Omega, F_3) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$.

By Lemma 2.1.4 a),b), the assertion follows from Proposition 1.4.10. ■

PROPOSITION 2.1.7 *Let*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$, Ω a locally compact space, Γ a closed set of Ω, $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, \mathbf{C}) \longrightarrow \mathscr{C}_0(\Omega, \mathbf{C})$ the inclusion map, and

$$\psi : \mathscr{C}_0(\Omega, \mathbf{C}) \longrightarrow \mathscr{C}_0(\Gamma, \mathbf{C}), \quad x \longmapsto x|\Gamma .$$

a) $G := \{\, x \in \mathscr{C}_0(\Omega, F_2) \mid x|\Gamma \in \mathscr{C}_0(\Gamma, F_1)\,\}$ *is a closed E-ideal of* $\mathscr{C}_0(\Omega, F_2)$*; we denote by* $\varphi' : G \longrightarrow \mathscr{C}_0(\Omega, F_2)$ *the inclusion map.*

b) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow G \xrightarrow{\varphi'} \mathscr{C}_0(\Omega, F_2) \xrightarrow{\phi_2 \otimes \psi} \mathscr{C}_0(\Gamma, F_3) \longrightarrow 0$$

is exact.

a) is easy to see.

b) We put

$$G_1 := \mathscr{C}_0(\Omega \setminus \Gamma, \mathbf{C}), \qquad G_2 := \mathscr{C}_0(\Omega, \mathbf{C}), \qquad G_3 := \mathscr{C}_0(\Gamma, \mathbf{C}) .$$

Let us consider the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes G_1 & \xrightarrow{\phi_1 \otimes id_{G_1}} & F_2 \otimes G_1 & \xrightarrow{\phi_2 \otimes id_{G_1}} & F_3 \otimes G_1 & \longrightarrow & 0 \\
 & & {\scriptstyle id_{F_1} \otimes \varphi}\downarrow & & {\scriptstyle id_{F_2} \otimes \varphi}\downarrow & & {\scriptstyle id_{F_3} \otimes \varphi}\downarrow & & \\
0 & \longrightarrow & F_1 \otimes G_2 & \xrightarrow{\phi_1 \otimes id_{G_2}} & F_2 \otimes G_2 & \xrightarrow{\phi_2 \otimes id_{G_2}} & F_3 \otimes G_2 & \longrightarrow & 0 \\
 & & {\scriptstyle id_{F_1} \otimes \psi}\downarrow & & {\scriptstyle id_{F_2} \otimes \psi}\downarrow & & {\scriptstyle id_{F_3} \otimes \psi}\downarrow & & \\
0 & \longrightarrow & F_1 \otimes G_3 & \xrightarrow{\phi_1 \otimes id_{G_3}} & F_2 \otimes G_3 & \xrightarrow{\phi_2 \otimes id_{G_3}} & F_3 \otimes G_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Lemma 2.1.4 a), Proposition 1.4.9, and Proposition 1.4.10, its columns and rows are exact. It follows that $\phi_2 \otimes \psi$ is surjective. Let $x \in Ker(\phi_2 \otimes \psi)$. Then

$$(id_{F_3} \otimes \psi)(\phi_2 \otimes id_{G_2})x = (\phi_2 \otimes \psi)x = 0,$$

so there is a $y \in F_2 \otimes G_1$ with

$$(\phi_2 \otimes \varphi)y = (id_{F_3} \otimes \varphi)(\phi_2 \otimes id_{G_1})y = (\phi_2 \otimes id_{G_2})x .$$

Then

$$(\phi_2 \otimes id_{G_2})(x - (id_{F_2} \otimes \varphi)y) = (\phi_2 \otimes id_{G_2})x - (\phi_2 \otimes \varphi)y = 0\,,$$

so there is a $z \in F_1 \otimes G_2$ with

$$(\phi_1 \otimes id_{G_2})z = x - (id_{F_2} \otimes \varphi)y\ .$$

Thus

$$x = (id_{F_2} \otimes \varphi)y + (\phi_1 \otimes id_{G_2})z \in G\,, \qquad Ker\,(\phi_2 \otimes \psi) \subset G\ .$$

Let now $x \in G$. By Proposition 1.4.9, there is a $y \in \mathscr{C}_0\,(\Omega, F_1) = F_1 \otimes G_2$ with $x|\Gamma = y|\Gamma$. There is a $z \in \mathscr{C}_0\,(\Omega \setminus \Gamma, F_2) = F_2 \otimes G_1$ with

$$(id_{F_2} \otimes \varphi)z = x - (\phi_1 \otimes id_{G_2})y\ .$$

We get

$$(\phi_2 \otimes \psi)x = (\phi_2 \otimes \psi)(\phi_1 \otimes id_{G_2})y + (\phi_2 \otimes \psi)(id_{F_2} \otimes \varphi)z =$$

$$= ((\phi_2 \circ \phi_1) \otimes \psi)y + (\phi_2 \otimes (\psi \circ \varphi))z = 0\,,$$

$G \subset Ker\,(\phi_2 \otimes \psi)$. ∎

Remark. If we put $F_1 := 0$ and $F_2 = F_3$ in the above Proposition then we obtain Tietze's Theorem (Corollary 2.1.5).

PROPOSITION 2.1.8 (Topological six-term sequence) *Let Ω be a locally compact space, Γ a closed set of Ω, $\varphi : \mathscr{C}_0\,(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0\,(\Omega, F)$ the inclusion map,*

$$\psi : \mathscr{C}_0\,(\Omega, F) \longrightarrow \mathscr{C}_0\,(\Gamma, F)\,, \quad x \longmapsto x|\Gamma\,,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$ (Tietze's Theorem (Corollary 2.1.5))

$$0 \longrightarrow \mathscr{C}_0\,(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0\,(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0\,(\Gamma, F) \longrightarrow 0\ .$$

a) Assume $\Omega \setminus \Gamma$ is Υ-null.

a_1) $K_i(\psi) : K_i(\mathscr{C}_0\,(\Omega, F)) \longrightarrow K_i(\mathscr{C}_0\,(\Gamma, F))$ is a group isomorphism.

a_2) If $\Omega \in \Upsilon$ or $\Gamma \in \Upsilon$ then

$$\Omega, \Gamma \in \Upsilon\,, \qquad p(\Omega) = p(\Gamma)\,, \qquad q(\Omega) = q(\Gamma)\,,$$

$$\Phi_{i,\Gamma,F} = K_i\,(id_F \otimes \psi) \circ \Phi_{i,\Omega,F}\,, \qquad \Omega_\Upsilon = \Gamma_\Upsilon\,.$$

b) Assume Ω is Υ-null.

b_1) $\delta_i : K_i(\mathscr{C}_0\,(\Gamma, F)) \longrightarrow K_{i+1}(\mathscr{C}_0\,(\Omega \setminus \Gamma, F))$ is a group isomorphism.

b_2) If $\Omega \setminus \Gamma \in \Upsilon$ or $\Gamma \in \Upsilon$ then

$$\Omega \setminus \Gamma, \Gamma \in \Upsilon\,, \qquad p(\Omega \setminus \Gamma) = q(\Gamma)\,, \qquad q(\Omega \setminus \Gamma) = p(\Gamma)\,,$$

$$\Phi_{i,\Gamma,F} = \Phi_{(i+1),(\Omega\setminus\Gamma),F} \circ \delta_i\,.$$

c) Assume Γ is Υ-null.

c_1) $K_i(\varphi) : K_i(\mathscr{C}_0\,(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}_0\,(\Omega, F))$ is a group isomorphism.

c_2) If $\Omega \setminus \Gamma \in \Upsilon$ or $\Omega \in \Upsilon$ then

$$\Omega \setminus \Gamma, \Omega \in \Upsilon\,, \qquad p(\Omega \setminus \Gamma) = p(\Omega)\,, \qquad q(\Omega \setminus \Gamma) = q(\Omega)\,,$$

$$\Phi_{i,\Omega,F} = K_i\,(id_F \otimes \varphi) \circ \Phi_{i,(\Omega\setminus\Gamma),F}\,, \qquad (\Omega \setminus \Gamma)_\Upsilon = \Omega_\Upsilon\,.$$

The assertions follow from Lemma 2.1.4 a),b) and Proposition 1.5.6. ■

COROLLARY 2.1.9 *Let Ω be a locally compact space, $\omega \in \Omega$ such that $\Omega \setminus \{\omega\}$ is Υ-null, Γ a closed set of Ω,*

$$\Omega' := (\Omega \setminus \{\omega\}) \setminus \Gamma\,, \qquad \Gamma' := \Gamma \setminus \{\omega\}\,,$$

$\varphi : \mathscr{C}_0\,(\Omega', F) \longrightarrow \mathscr{C}_0\,(\Omega \setminus \{\omega\}, F)$ the inclusion map,

$$\psi : \mathscr{C}_0\,(\Omega \setminus \{\omega\}, F) \longrightarrow \mathscr{C}_0\left(\Gamma', F\right)\,, \quad x \longmapsto x|\Gamma'\,,$$

and δ_i the index maps of the exact sequence in $\mathfrak{M}_E$ (Tietze's Theorem (Corollary 2.1.5))

$$0 \longrightarrow \mathscr{C}_0\left(\Omega', F\right) \xrightarrow{\varphi} \mathscr{C}_0\,(\Omega \setminus \{\omega\}, F) \xrightarrow{\psi} \mathscr{C}_0\left(\Gamma', F\right) \longrightarrow 0\,.$$

a) $\delta_i : K_i\,(\mathscr{C}_0\,(\Gamma', F)) \longrightarrow K_{i+1}\,(\mathscr{C}_0\,(\Omega', F))$ is a group isomorphism.

b) If $\Omega' \in \Upsilon$ or $\Gamma' \in \Upsilon$ then

$$\Omega', \Gamma' \in \Upsilon, \qquad p(\Omega') = q(\Gamma'), \qquad q(\Omega') = p(\Gamma'),$$

$$\Phi_{i,\Gamma',F} = \Phi_{(i+1),\Omega',F} \circ \delta_i .$$

c) If Γ is finite then

$$\Omega' \in \Upsilon, \qquad p(\Omega') = 0, \qquad q(\Omega') = Card\,\Gamma' .$$

a) and b) follow from the Topological six-term sequence (Proposition 2.1.8 b)).

c) follows from b) and Lemma 2.1.4 c). ■

COROLLARY 2.1.10 *Let Ω, Ω' be locally compact spaces, $\omega \in \Omega$, and $\omega' \in \Omega'$ such that $\Omega' \setminus \{\omega'\}$ is null-homotopic.*

a) $K_i(\mathscr{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F)) \approx K_i(\mathscr{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F))$.

b) If also $\Omega \setminus \{\omega\}$ is null-homotopic then $\mathscr{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F)$ is K-null.

a) The sequence in $\mathfrak{M}_E$ (with obvious notation)

$$0 \longrightarrow \mathscr{C}_0\left((\Omega \setminus \{\omega\}) \times \Omega', F\right) \xrightarrow{\varphi} \mathscr{C}_0\left((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F\right)$$

$$\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F\right) \xrightarrow{\psi} \mathscr{C}_0\left(\{\omega\} \times (\Omega' \setminus \{\omega'\}), F\right) \longrightarrow 0$$

is exact and the assertion follows from the Topological six-term sequence (Proposition 2.1.8 c_1)).

b) By Proposition 1.4.2 c) and Lemma 2.1.4 b), $(\Omega \setminus \{\omega\}) \times \Omega'$ is null-homotopic and so K-null (Proposition 1.5.4 a)). By a),

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F\right)\right)$$

is K-null. ■

PROPOSITION 2.1.11 (Topological triple) *Let Ω_1 be a locally compact space, Ω_2 an open set of Ω_1, Ω_3 an open set of Ω_2, and $\varphi : \mathscr{C}_0(\Omega_2 \setminus \Omega_3, F) \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)$ the inclusion map. For all $j,k \in \{1,2,3\}$, $j<k$, put*

$$\psi_{j,k} : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega_j \setminus \Omega_k, F)\,, \quad x \longmapsto x|(\Omega_j \setminus \Omega_k)$$

and denote by $\varphi_{j,k} : \mathscr{C}_0(\Omega_k, F) \longrightarrow \mathscr{C}_0(\Omega_j, F)$ the inclusion map and by $\delta_{j,k,i}$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega_k, F) \xrightarrow{\varphi_{j,k}} \mathscr{C}_0(\Omega_j, F) \xrightarrow{\psi_{j,k}} \mathscr{C}_0(\Omega_j \setminus \Omega_k, F) \longrightarrow 0\,.$$

a) Assume $\mathscr{C}_0(\Omega_2, F)$ K-null.

a_1) $\delta_{2,3,i} : K_i(\mathscr{C}_0(\Omega_2 \setminus \Omega_3, F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega_3, F))$ is a group isomorphism.

a_2) $\delta_{2,3,i} = \delta_{1,3,i} \circ K_i(\varphi)$.

a_3) $\varphi_{1,3}$ is K-null.

a_4) If we put $\Phi_i := K_i(\varphi) \circ (\delta_{2,3,i})^{-1}$ then

$$0 \longrightarrow K_i(\mathscr{C}_0(\Omega_1, F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)) \underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}} K_{i+1}(\mathscr{C}_0(\Omega_3, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(\mathscr{C}_0(\Omega_1, F)) \times K_{i+1}(\mathscr{C}_0(\Omega_3, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_1 \setminus \Omega_3, F))\,,$$

$$(a,b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

is a group isomorphism.

a_5) If Ω_2 is Υ-null and $\Omega_1, \Omega_3 \in \Upsilon$ then

$$\Omega_1 \setminus \Omega_3 \in \Upsilon\,,\; p(\Omega_1 \setminus \Omega_3) = p(\Omega_1) + q(\Omega_3)\,,\; q(\Omega_1 \setminus \Omega_3) = q(\Omega_1) + p(\Omega_3)\,,$$

and (with the notation of Proposition 1.5.13*)*

$$\Phi_{i,(\Omega_1 \setminus \Omega_3),F} = \Psi_{F,i} \circ (\Phi_{i,\Omega_1,F} \times \Phi_{(i+1),\Omega_3,F})\,.$$

b) Assume $\mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)$ K-null.

b_1) $\delta_{2,3,i} = 0$.

b_2) $K_i(\varphi_{1,3}) : K_i(\mathscr{C}_0(\Omega_3, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_1, F))$ is a group isomorphism.

b_3) If we put $\Phi_i := K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2})$ then the map

$$\Psi : K_i(\mathscr{C}_0(\Omega_2, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_3, F)) \times K_i(\mathscr{C}_0(\Omega_2 \setminus \Omega_3, F)),$$

$$b \longmapsto (\Phi_i b, K_i(\psi_{2,3})b)$$

is a group isomorphism.

b_4) If $\psi_{1,2}$ is K-null and if we put $\Phi'_i := K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1}$ (by c_2)) *then*

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega_1 \setminus \Omega_2, F)) \xrightarrow{\delta_{1,2,(i+1)}} K_i(\mathscr{C}_0(\Omega_2, F)) \underset{\Phi'_i}{\overset{K_i(\varphi_{1,2})}{\rightleftarrows}}$$

$$\underset{\Phi'_i}{\overset{K_i(\varphi_{1,2})}{\rightleftarrows}} K_i(\mathscr{C}_0(\Omega_1, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(\mathscr{C}_0(\Omega_1, F)) \times K_{i+1}(\mathscr{C}_0(\Omega_1 \setminus \Omega_2, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_2, F)),$$

$$(a, b) \longmapsto \Phi'_i a + \delta_{1,2,(i+1)} b$$

is a group isomorphism.

b_5) If $\Omega_1 \setminus \Omega_3$ is Υ-null, $\Omega_1, \Omega_1 \setminus \Omega_2 \in \Upsilon$, and $\psi_{1,2}$ is K-null then

$$\Omega_2 \in \Upsilon\,,\ p(\Omega_2) = p(\Omega_1) + q(\Omega_1 \setminus \Omega_2)\,,\ q(\Omega_2) = q(\Omega_1) + p(\Omega_1 \setminus \Omega_2)\,.$$

c) Assume $\mathscr{C}_0(\Omega_1, F)$ K-null and put

$$\psi : \mathscr{C}_0(\Omega_1 \setminus \Omega_3, F) \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Omega_2, F)\,, \quad x \longmapsto x|(\Omega_1 \setminus \Omega_2)\,.$$

c_1) $\delta_{1,2,i}$ and $\delta_{1,3,i}$ are group isomorphisms.

c_2) $K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} = \delta_{1,2,(i+1)} \circ K_{i+1}(\psi)$.

c_3) Let $\varphi' : \mathscr{C}_0(\Omega_1 \setminus \Omega_2, F) \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)$ be a morphism in $\mathfrak{M}_E$ such that

$$K_i(\psi \circ \varphi') = id_{K_i(\mathscr{C}_0(\Omega_1 \setminus \Omega_2, F))}\,.$$

If we put

$$\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi') \circ (\delta_{1,2,(i+1)})^{-1}$$

then $K_i(\varphi_{2,3})\circ\Phi_i = id_{K_i(\mathscr{C}_0(\Omega_2,F))}$. *If in addition* $\psi_{2,3}$ *is K-null then*

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega_2\setminus\Omega_3,F)) \xrightarrow{\delta_{2,3,(i+1)}} K_i(\mathscr{C}_0(\Omega_3,F)) \underset{\xleftarrow[\Phi_i]{}}{\xrightarrow{K_i(\varphi_{2,3})}}$$

$$\underset{\xleftarrow[\Phi_i]{}}{\xrightarrow{K_i(\varphi_{2,3})}} K_i(\mathscr{C}_0(\Omega_2,F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}((\mathscr{C}_0(\Omega_2\setminus\Omega_3,F))\times K_i(\mathscr{C}_0(\Omega_2,F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_3,F)),$$

$$(a,b)\longmapsto \delta_{2,3,(i+1)}a+\Phi_i b$$

is a group isomorphism.

Up to $a_5)$ and $b_5)$ the Proposition follows from Tietze's Theorem (Corollary 2.1.5) and from the triple theorem (Theorem 1.3.8) (and Lemma 2.1.4 a),b)). $a_5)$ follows from Proposition 1.5.13 and $b_5)$ follows from Proposition 1.5.14. ∎

COROLLARY 2.1.12 *Let* $F\overset{\phi}{\to}F'$ *be a morphism in* $\mathfrak{M}_E$. *We use the notation and hypotheses of* Proposition 2.1.11 *and the hypothesis that* $\mathscr{C}_0(\Omega_2,F)$ *and* $\mathscr{C}_0(\Omega_2,F')$ *are K-null, and mark with an accent those notation associated to* F'. *We put for all* $j\in\{1,2,3\}$ *and for all* $j,k\in\{1,2,3\}$, $j<k$,

$$\phi_j : \mathscr{C}_0(\Omega_j,F)\longrightarrow \mathscr{C}_0\left(\Omega_j,F'\right), \quad x\longmapsto \phi\circ x,$$

$$\phi_{j,k} : \mathscr{C}_0(\Omega_j\setminus\Omega_k,F)\longrightarrow \mathscr{C}_0\left(\Omega_j\setminus\Omega_k,F'\right), \quad x\longmapsto \phi\circ x.$$

a) $\Phi'_i\circ K_{i+1}(\phi_3) = K_i(\phi_{1,3})\circ\Phi_i$.

b) If we identify $K_i(\mathscr{C}_0(\Omega_1\setminus\Omega_3,F))$ *with* $K_i(\mathscr{C}_0(\Omega_1,F))\times K_{i+1}(\mathscr{C}_0(\Omega_3,F))$ *and* $K_i(\mathscr{C}_0(\Omega_1\setminus\Omega_3,F'))$ *with* $K_i(\mathscr{C}_0(\Omega_1,F'))\times K_{i+1}(\mathscr{C}_0(\Omega_3,F'))$ *using the isomorphisms of* Proposition 2.1.11 $a_4)$ *then*

$$K_i(\phi_{1,3}) : K_i(\mathscr{C}_0(\Omega_1\setminus\Omega_3,F))\longrightarrow K_i(\mathscr{C}_0\left(\Omega_1\setminus\Omega_3,F'\right)),$$

$$(a,b)\longrightarrow (K_i(\phi_1)a, K_{i+1}(\phi_3)b)$$

is a group isomorphism.

a) The diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_0(\Omega_3,F) \arrow[r, "\varphi_{2,3}"] \arrow[d, "\phi_3"] & \mathscr{C}_0(\Omega_2,F) \arrow[r, "\psi_{2,3}"] \arrow[d, "\phi_2"] & \mathscr{C}_0(\Omega_2\setminus\Omega_3,F) \arrow[r] \arrow[d, "\phi_{2,3}"] & 0 \\
0 \arrow[r] & \mathscr{C}_0(\Omega_3,F')) \arrow[r, "\varphi'_{2,3}"'] & \mathscr{C}_0(\Omega_2,F') \arrow[r, "\psi'_{2,3}"'] & \mathscr{C}_0(\Omega_2\setminus\Omega_3,F') \arrow[r] & 0
\end{tikzcd}$$

is obviously commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8),

$$K_{i+1}(\phi_3)\circ\delta_{2,3,i} = (\delta_{2,3,i})'\circ K_i(\phi_{2,3}),$$

$$\left((\delta_{2,3,i})'\right)^{-1}\circ K_{i+1}(\phi_3) = K_i(\phi_{2,3})\circ(\delta_{2,3,i})^{-1}\ .$$

By the above, since $\phi_{1,3}\circ\varphi = \varphi'\circ\phi_{2,3}$,

$$K_i(\phi_{1,3})\circ\Phi_i = K_i(\phi_{1,3})\circ K_i(\varphi)\circ(\delta_{2,3,i})^{-1} = K_i(\varphi')\circ K_i(\phi_{2,3})\circ(\delta_{2,3,i})^{-1} =$$

$$= K_i(\varphi')\circ\left((\delta_{2,3,i})'\right)^{-1}\circ K_{i+1}(\phi_3) = \Phi'_i\circ K_{i+1}(\phi_3)\ .$$

b) follows from a) and Proposition 2.1.11 a_4). ■

2.2 Alexandroff Compactification

THEOREM 2.2.1 (Alexandroff K-theorem) *Let Ω be a locally compact space and Ω^* its Alexandroff compactification. We denote by*

$$\varphi : \mathscr{C}_0(\Omega,F) \longrightarrow \mathscr{C}(\Omega^*,F)$$

the inclusion map and put

$$\lambda : F \longrightarrow \mathscr{C}(\Omega^*,F)\ , \quad y \longmapsto y1_{\mathscr{C}(\Omega^*,\mathbb{C})}$$

a) The map

$$K_i(\mathscr{C}_0(\Omega,F))\times K_i(F) \longrightarrow K_i(\mathscr{C}(\Omega^*,F)), \quad (a,b)\longmapsto K_i(\varphi)a+K_i(\lambda)b$$

is a group isomorphism.

b) If $\Omega \in \Upsilon$ *then*

$$\Omega^* \in \Upsilon, \qquad p(\Omega^*) = p(\Omega) + 1, \qquad q(\Omega^*) = q(\Omega) \qquad \Omega_\Upsilon \subset \Omega^*_\Upsilon .$$

c) Ω *is* Υ*-null iff* $\Omega^* \in \Upsilon_1$.

$\mathscr{C}(\Omega^*, \mathbf{C})$ is the unitization of $\mathscr{C}_0(\Omega, \mathbf{C})$.

a) Since

$$\mathscr{C}_0(\Omega, F) \approx F \otimes \mathscr{C}_0(\Omega, \mathbf{C}), \qquad \mathscr{C}_0(\Omega^*, F) \approx F \otimes \mathscr{C}_0(\Omega^*, \mathbf{C})$$

(Lemma 2.1.4 b)), the assertion follows from Corollary 1.4.5 b).

b) follows from Corollary 1.5.8.

c) follows from Proposition 1.6.7 b). ■

COROLLARY 2.2.2 *Let* Ω_1 *and* Ω_2 *be locally compact spaces,* Ω_1^*, Ω_2^* *their Alexandroff compactification, respectively,* $\vartheta : \Omega_1 \longrightarrow \Omega_2$ *a proper continuous map,* $\vartheta^* : \Omega_1^* \longrightarrow \Omega_2^*$ *its continuous extension, and*

$$\phi : \mathscr{C}_0(\Omega_2, F) \longrightarrow \mathscr{C}_0(\Omega_1, F), \quad x \longmapsto x \circ \vartheta,$$

$$\phi^* : \mathscr{C}(\Omega_2^*, F) \longrightarrow \mathscr{C}(\Omega_1^*, F), \quad x \longmapsto x \circ \vartheta^* .$$

a) If we identify $K_i\left(\mathscr{C}\left(\Omega_j^*, F\right)\right)$ *with* $K_i(\mathscr{C}_0(\Omega_j, F)) \times K_i(F)$ *for every* $j \in \{1,2\}$ *using the group isomorphisms of the Alexandroff K-theorem* (Theorem 2.2.1 a)) *then*

$$K_i(\phi^*) : K_i(\mathscr{C}(\Omega_2^*, F)) \longrightarrow K_i(\mathscr{C}(\Omega_1^*, F)), \quad (a,b) \longmapsto (K_i(\phi)a, b) .$$

b) Let $\vartheta' : \Omega_1 \longrightarrow \Omega_2$ *be a proper continuous map and let* ϕ', ϕ'^* *be the above maps associated to* ϑ'. *If* Ω_2 *is* Υ*-null then* $K_i(id_F \otimes \phi^*) = K_i(id_F \otimes \phi'^*)$. *In particular if* $\Omega_1 = \Omega_2$ *then*

$$K_i(id_F \otimes \phi^*) = id_{K_i(\mathscr{C}(\Omega_1^*, F))} .$$

a) follows from Corollary 1.4.5 c).

b) follows from Proposition 1.6.7 c). ∎

COROLLARY 2.2.3 *Let* $F \xrightarrow{\phi} F'$ *be a morphism in* $\mathfrak{M}_E$. *We use the notation of the Alexandroff K-theorem* (Theorem 2.2.1) *and put*

$$\phi_\Omega : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0\left(\Omega, F'\right), \quad x \longmapsto \phi \circ x,$$

$$\phi_{\Omega^*} : \mathscr{C}(\Omega^*, F) \longrightarrow \mathscr{C}\left(\Omega^*, F'\right), \quad x \longmapsto \phi \circ x.$$

If we identify $K_i(\mathscr{C}(\Omega^*, F))$ *with* $K_i(\mathscr{C}_0(\Omega, F)) \times K_i(F)$ *and* $K_i(\mathscr{C}(\Omega^*, F'))$ *with* $K_i(\mathscr{C}_0(\Omega, F')) \times K_i(F')$ *using the group isomorphism of the Alexandroff K-theorem* (Theorem 2.2.1 a)) *then*

$$K_i(\phi_{\Omega^*}) : K_i(\mathscr{C}(\Omega^*, F)) \longrightarrow K_i\left(\mathscr{C}\left(\Omega^*, F'\right)\right), \quad (a,b) \longmapsto (K_i(\phi_\Omega)a, K_i(\phi)b).$$

The assertion follows from Corollary 1.4.5 c). ∎

COROLLARY 2.2.4 *We use the notation of the Alexandroff K-theorem* (Theorem 2.2.1 a)) *and denote by* ω_∞ *the Alexandroff point of* Ω. *Let* Ω' *be a locally compact space,*

$$\varphi' : \mathscr{C}_0\left(\Omega \times \Omega', F\right) \longrightarrow \mathscr{C}_0\left(\Omega^* \times \Omega', F\right)$$

the inclusion map, and

$$\lambda' : \mathscr{C}_0\left(\Omega', F\right) \longrightarrow \mathscr{C}_0\left(\Omega^* \times \Omega', F\right), \quad x \longmapsto \tilde{x},$$

where

$$\tilde{x} : \Omega^* \times \Omega' \longrightarrow F, \quad (\omega, \omega') \longmapsto x(\omega').$$

Then the map

$$K_i\left(\mathscr{C}_0\left(\Omega \times \Omega', F\right)\right) \times K_i(F) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega^* \times \Omega', F\right)\right),$$

$$(a,b) \longmapsto K_i(\varphi')a + K_i(\lambda')b$$

is a group isomorphism.

If we put

$$\psi' : \mathscr{C}_0\left(\Omega^* \times \Omega', F\right) \longrightarrow \mathscr{C}_0\left(\Omega', F\right), \quad x \longmapsto x(\omega_\infty, \cdot)$$

then

$$0 \longrightarrow \mathscr{C}_0\left(\Omega \times \Omega', F\right) \xrightarrow{\varphi'} \mathscr{C}_0\left(\Omega^* \times \Omega', F\right) \underset{\overleftarrow{\lambda'}}{\overset{\psi'}{\rightarrow}} \mathscr{C}_0\left(\Omega', F\right) \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ and the assertion follows from the split exact axiom (Axiom 1.2.3). ■

2.3 Topological Sums of Locally Compact Spaces

PROPOSITION 2.3.1 (Product Theorem) *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces, Ω its topological sum, and for every $j \in J$ let $\varphi_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ be the inclusion map and*

$$\psi_j : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Omega_j, F), \quad x \longmapsto x|\Omega_j .$$

a)

$$\Phi_i : \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F)), \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} K_i(\varphi_j)a_j$$

is a group isomorphism and

$$\Psi_i : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j, F)), \quad a \longmapsto (K_i(\psi_j)a)_{j\in J}$$

is its inverse.

b) If all Ω_j, $j \in J$, belong to Υ then

$$\Omega \in \Upsilon, \qquad p(\Omega) = \sum_{j\in J} p(\Omega_j), \qquad q(\Omega) = \sum_{j\in J} q(\Omega_j),$$

$$\Phi_{i,\Omega,F} = \prod_{j\in J} \Phi_{i,\Omega_j,F}, \qquad \bigcap_{j\in J} (\Omega_j)_\Upsilon \subset \Omega_\Upsilon .$$

c) If Ω_j is Υ-null for every $j \in J$ then Ω is also Υ-null and $\Omega^* \in \Upsilon_1$, where Ω^* denotes the Alexandroff compactification of Ω.

a) follows from Proposition 1.3.3.

b) follows from Proposition 1.5.9.

c) By b), Ω is Υ-null and by Alexandroff's K-theorem (Theorem 2.2.1 a)), $\Omega^* \in \Upsilon_1$. ■

COROLLARY 2.3.2 *Let Ω be a locally compact space, Γ a closed set of Ω, and $(\Omega_j)_{j\in J}$ a finite family of pairwise disjoint open sets of Ω such that $\bigcup_{j\in J} \Omega_j = \Omega \setminus \Gamma$. We denote for every $j \in J$ by $\varphi_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ the inclusion map and assume that the maps*

$$K_i(\varphi_j) : K_i(\mathscr{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F))$$

are group isomorphisms. If $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ denotes the inclusion map and if we identify the above groups then $K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega, F))^J$ and

$$K_i(\varphi) : K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F)), \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} a_j \,. \qquad ■$$

COROLLARY 2.3.3 *Let Ω be a locally compact space such that $\mathscr{C}_0(\Omega, F)$ is K-null and Γ a closed set of Ω.*

a) $K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}(\Gamma, F))$.

b) Assume Γ finite and Ω Υ-null, put

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto x|\Gamma,$$

and denote by $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}(\Gamma, F) \longrightarrow 0 \,.$$

Then

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(F)^\Gamma,$$

$$\Omega \setminus \Gamma \in \Upsilon, \qquad p(\Omega \setminus \Gamma) = 0, \qquad q(\Omega \setminus \Gamma) = Card\,\Gamma, \qquad \Phi_{i,(\Omega\setminus\Gamma),F} = \delta_{i+1} \,.$$

a) Since $\mathscr{C}_0(\Omega, F)$ is K-null, the assertion follows from the six-term axiom (Axiom 1.2.7).

b) follows from a), Lemma 2.1.4 c), and the Product Theorem (Proposition 2.3.1). ■

COROLLARY 2.3.4 *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces, Ω its topological sum, and Ω^* the Alexandroff compactification of Ω.*

a) $K_i(\mathscr{C}(\Omega,F)) \approx \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F))\,,\ K_i(\mathscr{C}(\Omega^*,F)) \approx K_i(F)\times\prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F))\,.$

b) If all Ω_j, $j\in J$, belong to Υ then

$$\Omega^*\in\Upsilon\,, \qquad p(\Omega^*) = 1+\sum_{j\in J} p(\Omega_j)\,, \qquad q(\Omega^*) = \sum_{j\in J} q(\Omega_j)\,.$$

The assertion follows immediately from the Product Theorem (Proposition 2.3.1 a)) and the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

COROLLARY 2.3.5 *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces such that $\mathscr{C}_0(\Omega_j,F)$ is K-null for every $j\in J$ and let Γ_j be a closed set of Ω_j for every $j\in J$. We denote by Ω the Alexandroff compactification of the topological sum of the family $(\Omega_j\setminus\Gamma_j)_{j\in J}$.*

a) $K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times\prod_{j\in J} K_{i+1}(\mathscr{C}_0(\Gamma_j,F))$.

b) If for every $j\in J$, Ω_j is Υ-null and Γ_j is finite then

$$\Omega\in\Upsilon\,, \qquad p(\Omega)=1\,, \qquad q(\Omega) = \sum_{j\in J} Card\,\Gamma_j\,.$$

a) By Corollary 2.3.3 a), $K_i(\mathscr{C}_0(\Omega_j\setminus\Gamma_j,F)) \approx K_{i+1}(\mathscr{C}_0(\Gamma_j,F))$ for every $j\in J$ so by Corollary 2.3.4 a),

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times\prod_{j\in J} K_{i+1}(\mathscr{C}_0(\Gamma_j,F))\,.$$

b) By Corollary 2.3.3 b), for every $j\in J$,

$$\Omega_j\setminus\Gamma_j\in\Upsilon\,, \qquad p(\Omega_j\setminus\Gamma_j)=0\,, \qquad q(\Omega_j\setminus\Gamma_j) = Card\,\Gamma_j\,.$$

Thus by Corollary 2.3.4 b),

$$\Omega\in\Upsilon\,, \qquad p(\Omega)=1\,, \qquad q(\Omega) = \sum_{j\in J} Card\,\Gamma_j\,. \qquad ■$$

PROPOSITION 2.3.6 *Let Ω be a compact space belonging to Υ_1, Γ a closed set of Ω, $\omega_0 \in \Gamma$, and $\Gamma' := \Gamma \setminus \{\omega_0\}$. We use the notation of* the Topological triple (Proposition 2.1.11) *and put there*

$$\Omega_1 := \Omega\,, \qquad \Omega_2 := \Omega \setminus \{\omega_0\}\,, \qquad \Omega_3 := \Omega \setminus \Gamma\,.$$

a) $\Omega \setminus \{\omega_0\}$ *is Υ-null.*

b) $K_i\left(\mathscr{C}_0\left(\Omega \setminus \Gamma, F\right)\right) \approx K_{i+1}\left(\mathscr{C}_0\left(\Gamma', F\right)\right)$.

c)

$$0 \longrightarrow K_i(\mathscr{C}(\Omega, F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}(\Gamma, F)) \underset{\xleftarrow{\Phi_i}}{\xrightarrow{\delta_{1,3,i}}}$$

$$\underset{\xleftarrow{\Phi_i}}{\xrightarrow{\delta_{1,3,i}}} K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow 0$$

is a split exact sequence, and the maps

$$K_i(\mathscr{C}(\Omega, F)) \times K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b\,,$$

$$\delta_{2,3,i} : K_i\left(\mathscr{C}_0\left(\Gamma', F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F))$$

are group isomorphisms.

d) If $\Omega \setminus \Gamma \in \Upsilon$ or $\Gamma' \in \Upsilon$ then with the notation of Corollary 2.1.9

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Gamma', F\right)\right) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\Omega \setminus \{\omega_0\}, F\right)\right)$$

is a group isomorphism and

$$\Omega \setminus \Gamma, \Gamma' \in \Upsilon\,, \qquad p(\Omega \setminus \Gamma) = q(\Gamma')\,, \qquad q(\Omega \setminus \Gamma) = p(\Gamma')\,,$$

$$\Phi_{i,(\Omega \setminus \Gamma),F} = \delta_{i+1} \circ \Phi_{(i+1),\Gamma',F}\,.$$

e) Assume Γ finite.

e$_1$) $(\delta_{2,3,i})^{-1} : K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(F)^{\Gamma'}$ *is a group isomorphism.*

e$_2$) $\Omega \setminus \Gamma \in \Upsilon\,, \qquad p(\Omega \setminus \Gamma) = 0\,, \qquad q(\Omega \setminus \Gamma) = Card\, \Gamma'\,.$

a) follows from Alexandroff's K-theorem (Theorem 2.2.1 c)).

b) follows from Corollary 2.3.3 a).

c) By a), $\Omega \setminus \{\omega\}$ is K-null and the assertion follows from the Topological triple (Proposition 2.1.11 a)).

d) follows from Corollary 2.1.9.

$e_1)$ follows from c) and the Product Theorem (Proposition 2.3.1 a_4)).

$e_2)$ follows from a) and Corollary 2.1.9 c). ■

PROPOSITION 2.3.7 *Let Ω be a locally compact space, Γ a closed set of Ω, $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ the inclusion map,*

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Gamma, F)\,, \quad x \longmapsto x|\Gamma,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0(\Gamma, F) \longrightarrow 0\,.$$

Let $(\Omega_j)_{j \in J}$ be a finite family of pairwise disjoint open sets of Ω the union of which is $\Omega \setminus \Gamma$ and for every $j \in J$ put

$$\psi_j : \mathscr{C}_0\left(\bar{\Omega}_j, F\right) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}_j \setminus \Omega_j, F\right)\,, \quad x \longmapsto x|(\bar{\Omega}_j \setminus \Omega_j)\,,$$

$$\psi'_j : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega_j, F)\,, \quad x \longmapsto x|\Omega_j\,,$$

$$\psi''_j : \mathscr{C}_0(\Gamma, F) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}_j \setminus \Omega_j, F\right)\,, \quad x \longmapsto x|(\bar{\Omega}_j \setminus \Omega_j)$$

and denote by

$$\varphi_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}_j, F\right)\,,$$

$$\varphi'_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F)\,,$$

$$\varphi''_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$$

the inclusion maps and by $\delta_{j,i}$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega_j, F) \xrightarrow{\varphi_j} \mathscr{C}_0\left(\bar{\Omega}_j, F\right) \xrightarrow{\psi_j} \mathscr{C}_0\left(\bar{\Omega}_j \setminus \Omega_j, F\right) \longrightarrow 0\,.$$

a) For every $j \in J$,

$$\delta_{j,i} \circ K_i(\psi''_j) = K_{i+1}(\psi'_j) \circ \delta_i$$

and

$$\delta_i = \sum_{j \in J} K_{i+1}(\varphi'_j) \circ \delta_{j,i} \circ K_i(\psi''_j) \, .$$

b) $K_i(\varphi) = \sum_{j \in J} K_i(\varphi''_j) \circ K_i(\psi'_j)$.

c) Let $j_0 \in J$ *such that* $\mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right)$ *is K-null.*

c$_1$*)* $K_i(\varphi''_{j_0})$ *is a group isomorphism.*

c$_2$*) Assume* ψ *K-null. If we put*

$$\Phi_i := K_i(\varphi'_{j_0}) \circ K_i(\varphi''_{j_0})^{-1} : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F))$$

then

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Gamma, F)) \xrightarrow{\delta_{i+1}} K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \underset{\Phi_i}{\overset{K_i(\varphi)}{\rightleftarrows}} K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}(\mathscr{C}_0(\Gamma, F)) \times K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)),$$

$$(a,b) \longmapsto \delta_{i+1} a + \Phi_i b$$

is a group isomorphism.

a) By the commutativity of the index maps (Axiom 1.2.8),

$$\delta_{j,i} \circ K_i(\psi''_j) = K_{i+1}(\psi'_j) \circ \delta_i \, .$$

Since $\sum_{j \in J} \varphi'_j \circ \psi'_j$ is the identity map of $\mathscr{C}_0(\Omega \setminus \Gamma, F)$,

$$\sum_{j \in J} K_{i+1}(\varphi'_j) \circ \delta_{j,i} \circ K_i(\psi''_j) = \sum_{j \in J} K_{i+1}(\varphi'_j) \circ K_{i+1}(\psi'_j) \circ \delta_i =$$

$$= K_{i+1}\left(\sum_{j \in J} \varphi'_j \circ \psi'_j\right) \circ \delta_i = \delta_i \, .$$

b) We have $\varphi''_j = \varphi \circ \varphi'_j$ for every $j \in J$. Since $\sum\limits_{j \in J} \varphi'_j \circ \psi'_j$ is the identity map of $\mathscr{C}_0(\Omega \setminus \Gamma, F)$,

$$K_i(\varphi) = K_i(\varphi) \circ K_i\left(\sum_{j \in J} \varphi'_j \circ \psi'_j\right) =$$

$$= \sum_{j \in J} K_i(\varphi) \circ K_i(\varphi'_j) \circ K_i(\psi'_j) = \sum_{j \in J} K_i(\varphi''_j) \circ K_i(\psi'_j) \,.$$

c_1) If we put

$$\bar{\psi} : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right), \quad x \longmapsto x|(\Omega \setminus \Omega_{j_0})$$

then

$$0 \longrightarrow \mathscr{C}_0\left(\Omega_{j_0}, F\right) \xrightarrow{\varphi''_{j_0}} \mathscr{C}_0(\Omega, F) \xrightarrow{\bar{\psi}} \mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. Since $\mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right)$ is K-null, it follows that $K_i(\varphi''_{j_0})$ is a group isomorphism by the Topological six-term sequence (Proposition 2.1.8 c_1)).

c_2) Since $\varphi \circ \varphi'_{j_0} = \varphi''_{j_0}$,

$$K_i(\varphi) \circ \Phi_i = K_i(\varphi) \circ K_i(\varphi'_{j_0}) \circ K_i(\varphi''_{j_0})^{-1} = K_i(\varphi''_{j_0}) \circ K_i(\varphi''_{j_0})^{-1} = id_{K_i(\mathscr{C}_0(\Omega, F))} \,.$$

Since ψ is K-null,

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Gamma, F)) \xrightarrow{\delta_{i+1}} K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \underset{\xleftarrow[\Phi_i]{}}{\xrightarrow{K_i(\varphi)}} K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow 0$$

is a split exact sequence and this implies the last assertion. ■

PROPOSITION 2.3.8 *If $(\Omega_j)_{j \in J}$, $J \neq \emptyset$, is a finite family of compact spaces belonging to Υ_1 then $\prod\limits_{j \in J} \Omega_j \in \Upsilon_1$.*

The assertion follows immediately from Proposition 1.6.5. ■

2.4 Homotopy

PROPOSITION 2.4.1 *Let Ω be a locally compact space, Ω^* its Alexandroff compactification, $(\vartheta_s)_{s \in]0,1]}$ a family of proper continuous maps $\Omega \to \Omega$, and for every $s \in]0,1]$ let $\vartheta_s^* : \Omega^* \longrightarrow \Omega^*$ be the continuous extension of ϑ_s. We assume:*

1) $\Omega^* \times]0,1] \longrightarrow \Omega^*, \quad (\omega, s) \longmapsto \vartheta_s^*(\omega)$ *is continuous,*

2) $\vartheta_1(\omega) = \omega$ *for every* $\omega \in \Omega$,

3) for every compact set Γ *of* Ω *there is an* $\varepsilon \in]0,1]$ *with* $\Gamma \cap \vartheta_s(\Omega) = \emptyset$ *for all* $s \in]0, \varepsilon[$.

Then Ω *is null-homotopic and* $\Omega^* \in \Upsilon_1$.

We put for every $s \in [0,1]$,

$$\phi_s : \mathscr{C}_0(\Omega, \mathbb{C}) \longrightarrow \mathscr{C}_0(\Omega, \mathbb{C}), \quad x \longmapsto \begin{cases} x \circ \vartheta_s & \text{if} \quad s \in]0,1] \\ 0 & \text{if} \quad s = 0 \end{cases} .$$

Then $(\phi_s)_{s \in [0,1]}$ is a pointwise continuous path in $\mathscr{C}_0(\Omega, \mathbb{C})$ with $\phi_0 = 0$ and ϕ_1 the identity map of $\mathscr{C}_0(\Omega, F)$. Thus Ω is null-homotopic. By Proposition 1.5.4 d), Ω is Υ-null and by Alexandroff's K-theorem, (Theorem 2.2.1 c)), $\Omega^* \in \Upsilon_1$. ■

COROLLARY 2.4.2 *Let* J *be a set and* $\Omega := [0,1]^J$. *Then* $\Omega \setminus \{0\}$ *is null-homotopic and* $\Omega \in \Upsilon_1$.

The assertion follows from Proposition 2.4.1 by using the map

$$\vartheta : \Omega \times [0,1] \longrightarrow \Omega, \quad (\omega, s) \longmapsto s\omega .$$ ■

PROPOSITION 2.4.3 *Let* Ω *be a locally compact space,* Γ_0, Γ_1 *compact subspaces of* Ω, $\vartheta_0 : \Gamma_0 \longrightarrow \Gamma_1$ *a homeomorphism, and* $\vartheta : \Gamma_0 \times [0,1] \longrightarrow \Omega$ *a continuous map such that* $\vartheta(\omega, 0) = \omega$ *and* $\vartheta(\omega, 1) = \vartheta_0(\omega)$ *for every* $\omega \in \Gamma_0$. *We put*

$$\psi_j : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\Gamma_j, F), \quad x \longmapsto x|\Gamma_j$$

for every $j \in \{0,1\}$ *and*

$$\varphi : \mathscr{C}(\Gamma_1, F) \longrightarrow \mathscr{C}(\Gamma_0, F), \quad x \longmapsto x \circ \vartheta_0 .$$

a) $K_i(\varphi)$ *is a group isomorphism and* $K_i(\psi_0) = K_i(\varphi) \circ K_i(\psi_1)$.

b) For every $j \in \{0,1\}$ let $\varphi_j : \mathscr{C}_0(\Omega \setminus \Gamma_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ be the inclusion map and $\mathscr{C}(\Gamma_j, F) \overset{\lambda_j}{\to} \mathscr{C}_0(\Omega, F)$ be a morphism in $\mathfrak{M}_E$ such that $\psi_j \circ \lambda_j = id_{\mathscr{C}(\Gamma_j,F)}$ and $\lambda_1 = \lambda_0 \circ \varphi$.

b_1) For every $j \in \{0,1\}$,

$$0 \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) \xrightarrow{K_i(\varphi_j)} K_i(\mathscr{C}_0(\Omega, F)) \underset{\xleftarrow[K_i(\lambda_j)]{}}{\xrightarrow{K_i(\psi_j)}}$$

$$\underset{\xleftarrow[K_i(\lambda_j)]{}}{\xrightarrow{K_i(\psi_j)}} K_i(\mathscr{C}_0(\Gamma_j, F)) \longrightarrow 0$$

is a split exact sequence.

b_2) $Im K_i(\varphi_0) = Im K_i(\varphi_1)$.

b_3) If we put for every $j \in \{0,1\}$

$$\Psi_{j,i} : K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) \longrightarrow Im K_i(\varphi_j), \quad a \longmapsto K_i(\varphi_j) a$$

then $\Psi_{j,i}$ and

$$(\Psi_{1,i})^{-1} \circ \Psi_{0,i} : K_i(\mathscr{C}_0(\Omega \setminus \Gamma_0, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma_1, F))$$

are well-defined group isomorphisms.

b_4) If $\Omega \setminus \Gamma_0 \in \Upsilon$ or $\Omega \setminus \Gamma_1 \in \Upsilon$ then

$$\Omega \setminus \Gamma_0, \Omega \setminus \Gamma_1 \in \Upsilon, \qquad p(\Omega \setminus \Gamma_0) = p(\Omega \setminus \Gamma_1), \qquad q(\Omega \setminus \Gamma_0) = q(\Omega \setminus \Gamma_1),$$

$$\Phi_{i,(\Omega \setminus \Gamma_1),F} = (\Psi_{1,i}^F)^{-1} \circ \Psi_{0,i}^F \circ \Phi_{i,(\Omega \setminus \Gamma),F}.$$

c) If Ω is compact and if for every $j \in \{0,1\}$ there is a continuous map $\vartheta'_j : \Omega \longrightarrow \Gamma_j$ such that $\vartheta'_j(\omega) = \omega$ for every $\omega \in \Gamma_j$ and $\vartheta_0 \circ \vartheta'_0 = \vartheta'_1$ then the hypotheses of b) are fulfilled.

a) For every $s \in [0,1]$ put

$$\nu_s : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\Gamma_0, F), \quad x \longmapsto x(\vartheta(\cdot, s)).$$

Then $K_i(\nu_0) = K_i(\nu_1)$ by the homotopy axiom (Axiom 1.2.5). $K_i(\varphi)$ is obviously a group isomorphism. For every $x \in \mathscr{C}_0(\Omega, F)$ and $\omega \in \Gamma_0$,

$$(\nu_0 x)(\omega) = x(\vartheta(\omega, 0)) = x(\omega) = (\psi_0 x)(\omega),$$

$$(\nu_1 x)(\omega) = x(\vartheta(\omega,1)) = x(\vartheta_0(\omega)) = (\psi_1 x)(\vartheta_0(\omega)) = (\varphi\psi_1 x)(\omega),$$

so $\nu_0 = \psi_0$, $\nu_1 = \varphi \circ \psi_1$,

$$K_i(\psi_0) = K_i(\nu_0) = K_i(\nu_1) = K_i(\varphi) \circ K_i(\psi_1) .$$

b_1 follows from the split exact axiom (Axiom 1.2.3).

b_2) Let $j \in \{0,1\}$. We want to prove

$$Im K_i(\varphi_j) = \left\{ c - K_i(\lambda_j) K_i(\psi_j) c \;\middle|\; c \in K_i(\mathscr{C}(\Omega,F)) \right\} .$$

Let $a \in K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F))$ and put $c := K_i(\varphi_j) a$. Then

$$c - K_i(\lambda_j) K_i(\psi_j) c = K_i(\varphi_j) a - K_i(\lambda_j) K_i(\psi_j) K_i(\varphi_j) a = K_i(\varphi_j) a,$$

which proves the "$\subset$ "-inclusion. Let $c \in K_i(\mathscr{C}(\Omega,F))$. Then

$$K_i(\psi_j)(c - K_i(\lambda_j) K_i(\psi_j) c) =$$

$$= K_i(\psi_j) c - K_i(\psi_j) K_i(\lambda_j) K_i(\psi_j) c = K_i(\psi_j) c - K_i(\psi_j) c = 0,$$

$$c - K_i(\lambda_j) K_i(\psi_j) c \in Ker K_i(\psi_j) = Im K_i(\varphi_j) ,$$

which proves the "$\supset$"-inclusion (by b_1)).

Since $\lambda_1 \circ \psi_1 = \lambda_0 \circ \varphi \circ \psi_1$, we get by a),

$$K_i(\lambda_1) \circ K_i(\psi_1) = K_i(\lambda_0) \circ K_i(\varphi) \circ K_i(\psi_1) = K_i(\lambda_0) \circ K_i(\psi_0) .$$

Thus, by the above, $Im K_i(\varphi_0) = Im K_i(\varphi_1)$.

b_3) By b_1), $K_i(\varphi_0)$ and $K_i(\varphi_1)$ are injective, the assertion follows from b_2).

b_4) Let $F \xrightarrow{\phi} F'$ be o morphism in $\mathfrak{M}_E$ and for every $j \in \{0,1\}$ put

$$\mu_j : \mathscr{C}_0(\Omega \setminus \Gamma_j, F) \longrightarrow \mathscr{C}_0\left(\Omega \setminus \Gamma_j, F'\right) , \quad x \longmapsto \phi \circ x,$$

$$\mu : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0\left(\Omega, F'\right) , \quad x \longmapsto \phi \circ x .$$

We mark by a prime the notation associated to F when applied to F'. For every $j \in \{0,1\}$ the diagram

$$\begin{array}{ccc} \mathscr{C}_0(\Omega \setminus \Gamma_j, F) & \xrightarrow{\mu_j} & \mathscr{C}_0(\Omega \setminus \Gamma_j, F') \\ {\scriptstyle \varphi_j}\downarrow & & \downarrow{\scriptstyle \varphi'_j} \\ \mathscr{C}_0(\Omega, F) & \xrightarrow[\mu]{} & \mathscr{C}_0(\Omega, F') \end{array}$$

is commutative. Thus the diagrams

$$\begin{array}{ccc} K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) & \xrightarrow{K_i(\mu_j)} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F')) \\ {\scriptstyle K_i(\varphi_j)}\downarrow & & \downarrow{\scriptstyle K_i(\varphi'_j)} \\ K_i(\mathscr{C}_0(\Omega, F)) & \xrightarrow[K_i(\mu)]{} & K_i(\mathscr{C}_0(\Omega, F')) \end{array}$$

$$\begin{array}{ccc} K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) & \xrightarrow{K_i(\mu_j)} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F')) \\ {\scriptstyle \Psi_{j,i}}\downarrow & & \downarrow{\scriptstyle \Psi'_{j,i}} \\ Im K_i(\varphi_j) & \xrightarrow[\Lambda_i]{} & Im K_i\left(\varphi'_j\right) \end{array}$$

are also commutative, where Λ_i is the map defined by $K_i(\mu)$.

Assume $\Omega \setminus \Gamma_0 \in \Upsilon$ and consider the diagram (by b_2))

$$\begin{array}{ccc} K_i(F)^{p(\Omega\setminus\Gamma_0)} \times K_{i+1}(F)^{q(\Omega\setminus\Gamma_0)} & \xrightarrow{\Delta} & A \\ {\scriptstyle \Phi_{i,(\Omega\setminus\Gamma_0),F}}\downarrow & & \downarrow{\scriptstyle \Phi_{i,(\Omega\setminus\Gamma_0),F'}} \\ K_i(\mathscr{C}_0(\Omega \setminus \Gamma_0, F)) & \xrightarrow{K_i(\mu_0)} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_0, F')) \\ {\scriptstyle \Psi_{0,i}}\downarrow & & \downarrow{\scriptstyle \Psi'_{0,i}} \\ Im K_i(\varphi_0) & \xrightarrow[\Lambda_i]{} & Im K_i(\varphi'_0) \\ {\scriptstyle \Psi_{1,i}}\uparrow & & \uparrow{\scriptstyle \Psi'_{1,i}} \\ K_i(\mathscr{C}_0(\Omega \setminus \Gamma_1, F)) & \xrightarrow[K_i(\mu_1)]{} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_1, F')) \end{array}$$

where

$$\Delta := K_i(\phi)^{p(\Omega\setminus\Gamma_0)} \times K_{i+1}(\phi)^{q(\Omega\setminus\Gamma_0)} \,,$$

$$A := K_i\left(F'\right)^{p(\Omega\setminus\Gamma_0)} \times K_{i+1}\left(F'\right)^{q(\Omega\setminus\Gamma_0)} \,.$$

By the above, this diagram is commutative and the assertion follows from b_3).

c) For every $j \in \{0,1\}$ put

$$\lambda_j : \mathscr{C}(\Gamma_j, F) \longrightarrow \mathscr{C}(\Omega, F), \quad x \longmapsto x \circ \vartheta'_j .$$

Then $\psi_j \circ \lambda_j = id_{\mathscr{C}(\Gamma_j, F)}$ and for every $x \in \mathscr{C}(\Gamma_1, F)$,

$$\lambda_1 x = x \circ \vartheta'_1 = x \circ \vartheta_0 \circ \vartheta'_0 = (\varphi x) \circ \vartheta'_0 = \lambda_0(\varphi x), \qquad \lambda_1 = \lambda_0 \circ \varphi . \qquad \blacksquare$$

COROLLARY 2.4.4 *Let Ω be a compact space and $\omega, \omega' \in \Omega$ such that there is a continuous path in Ω from ω to ω'.*

a) $K_i(\mathscr{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i(\mathscr{C}_0(\Omega \setminus \{\omega'\}, F))$.

b) If $\Omega \setminus \{\omega\} \in \Upsilon$ *then*

$$\Omega \setminus \{\omega'\} \in \Upsilon, \qquad p(\Omega \setminus \{\omega'\}) = p(\Omega \setminus \{\omega\}), \qquad q(\Omega \setminus \{\omega'\}) = q(\Omega \setminus \{\omega\}) .$$

a) follows from Proposition 2.4.3 b_3) and c).

b) follows from Proposition 2.4.3 b_4) and c). $\blacksquare$

COROLLARY 2.4.5 *Let Ω, Ω' be compact spaces such that $\Omega' \setminus \{\omega'\}$ is null-homotopic for all $\omega' \in \Omega'$, $\omega \in \Omega$, and $\omega'' \in \Omega \times \Omega'$. Then*

$$K_i(\mathscr{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega\}) \times \Omega', F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \times \Omega' \setminus \{\omega''\}, F\right)\right) .$$

Let $\omega'' =: (\omega_0, \omega'_0) \in \Omega \times \Omega'$. By Corollary 2.1.10 a),

$$K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega_0\}) \times \Omega', F\right)\right) \approx K_i\left(\mathscr{C}_0\left(\Omega \times \Omega' \setminus \{\omega''\}, F\right)\right)$$

and by Proposition 2.4.3 c),

$$K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega\}) \times \Omega', F\right)\right) \approx K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega_0\}) \times \Omega', F\right)\right) .$$

By Proposition 1.4.2 b_3),c),

$$\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times(\Omega'\setminus\{\omega_0'\}),F\right)\approx\mathscr{C}_0\left(\Omega'\setminus\{\omega_0'\},\mathbb{C}\right)\otimes\mathscr{C}_0\left(\Omega\setminus\{\omega\},F\right)$$

is null-homotopic. Since the sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times(\Omega'\setminus\{\omega_0'\}),F\right)\longrightarrow\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\Omega',F\right)$$

$$\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\Omega',F\right)\longrightarrow\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\{\omega_0'\},F\right)\longrightarrow 0$$

is exact it follows from the topological six-term sequence (Proposition 2.1.8 a_1)),

$$K_i\left(\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\Omega',F\right)\right)\approx$$

$$\approx K_i\left(\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\{\omega_0'\},F\right)\right)\approx K_i\left(\mathscr{C}_0\left(\Omega\setminus\{\omega\},F\right)\right)\,.$$ ■

COROLLARY 2.4.6 *Let Ω be a locally compact space and $\omega_1,\omega_2\in\Omega$ and for every $j\in\{1,2\}$ put*

$$\psi_j:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow F\,,\quad x\longmapsto x(\omega_j)\,.$$

If there is a continuous path in Ω from ω_1 to ω_2 then $K_i\left(\psi_1\right)=K_i\left(\psi_2\right)$.

The assertion follows from Proposition 2.4.3 a). ■

COROLLARY 2.4.7 *Let Ω be a locally compact space, Γ a finite subset of Ω, $\omega_0\in\Omega$, and*

$$\psi:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow\mathscr{C}\left(\Gamma,F\right),\quad x\longmapsto x|\Gamma,$$

$$\psi_{\omega_0}:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow F\,,\quad x\longmapsto x(\omega_0)\,.$$

If for every $\omega\in\Gamma$ there is a continuous path in Ω connecting ω_0 with ω then

$$K_i(\psi):K_i(\mathscr{C}_0\left(\Omega,F\right))\longrightarrow K_i(\mathscr{C}\left(\Gamma,F\right))\approx K_i(F)^{Card\,\Gamma},$$

$$a\longmapsto\left(K_i(\psi_{\omega_0})a\right)_{\omega\in\Gamma}\,.$$

We put

$$\psi_\omega:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow F\,,\quad x\longmapsto x(\omega)$$

for every $\omega\in\Gamma$. By Corollary 2.4.6 , $K_i(\psi_\omega)=K_i(\psi_{\omega_0})$ for every $\omega\in\Gamma$ and the assertion follows from the Product Theorem (Proposition 2.3.1 a)). ■

PROPOSITION 2.4.8 *Let Ω be a path connected compact space, Γ a finite subset of Ω, $\omega_0 \in \Gamma$, $\Gamma' := \Gamma \setminus \{\omega_0\}$,*

$$\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}(\Omega, F)\,,$$

$$\varphi' : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega \setminus \{\omega_0\}, F)\,,$$

$$\varphi'' : \mathscr{C}\left(\Gamma', F\right) \longrightarrow \mathscr{C}(\Gamma, F)$$

the inclusion maps,

$$\psi : \mathscr{C}(\Omega, F) \longrightarrow \mathscr{C}(\Gamma, F)\,, \quad x \longmapsto x|\Gamma,$$

$$\psi' : \mathscr{C}_0(\Omega \setminus \{\omega_0\}, F) \longrightarrow \mathscr{C}\left(\Gamma', F\right)\,, \quad x \longmapsto x|\Gamma'\,,$$

$$\psi_\omega : \mathscr{C}(\Omega, F) \longrightarrow F\,, \quad x \longmapsto x(\omega)$$

for every $\omega \in \Gamma$, and δ_i, δ'_i the index maps associated to the exact sequences in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}(\Omega, F) \xrightarrow{\psi} \mathscr{C}(\Gamma, F) \longrightarrow 0\,,$$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi'} \mathscr{C}_0(\Omega \setminus \{\omega_0\}, F) \xrightarrow{\psi'} \mathscr{C}\left(\Gamma', F\right) \longrightarrow 0\,.$$

a) $K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times K_i(\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F))$.

b) ψ' is K-null.

c) If we use the group isomorphism of a) then

$$K_i(\psi) : K_i(\mathscr{C}(\Omega, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F)^\Gamma\,, \quad (a, b) \longmapsto (a)_{\omega \in \Gamma}\,.$$

d) If we identify $K_i(\mathscr{C}(\Gamma, F))$ with $K_i(F)^\Gamma$ and $K_i(\mathscr{C}(\Gamma', F))$ with $K_i(F)^{\Gamma'}$ then

$$\delta_i : K_i(\mathscr{C}(\Gamma, \cdot)) \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, \cdot))\,, \quad (a_\omega)_{\omega \in \Gamma} \longmapsto (\delta'_i(a_\omega - a_{\omega_0}))_{\omega \in \Gamma'}\,.$$

e) Assume $\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F)$ K-null.

e_1) $K_i(\psi_{\omega_0}) : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(F)$ is a group isomorphism.

e_2) $\delta'_i : K_i(\mathscr{C}(\Gamma', F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F))$ is a group isomorphism.

e_3) If we identify $K_i(\mathscr{C}(\Gamma', F))$ with $K_i(F)^{\Gamma'}$ and $K_i(\mathscr{C}(\Gamma, F))$ with $K_i(F)^\Gamma$ then for all $(a_\omega)_{\omega \in \Gamma'}$

$$K_i(\varphi'')(a_\omega)_{\omega \in \Gamma'} = (a_\omega)_{\omega \in \Gamma}\,,$$

where $a_{\omega_0} = 0$.

e_4) If we identify $K_{i+1}(\mathscr{C}_0(\Omega\setminus\Gamma,F))$ with $K_i(\mathscr{C}(\Gamma',F))$ using $(\delta_i')^{-1}$ of e_2) then for all $(a_\omega)_{\omega\in\Gamma}\in K_i(\mathscr{C}(\Gamma,F))$,

$$\delta_i(a_\omega)_{\omega\in\Gamma}=(a_\omega-a_{\omega_0})_{\omega\in\Gamma'}\,.$$

a) follows from the Alexandroff K-theorem (Theorem 2.2.1 a)).

b) Let $\omega\in\Gamma'$ and let $\vartheta:[0,1]\longrightarrow\Omega$ be a continuous path in Ω connecting ω with ω_0. Then for every $x\in\mathscr{C}_0(\Omega\setminus\{\omega_0\},F)$ the map

$$[0,1]\longrightarrow\mathscr{C}_0(\Omega\setminus\{\omega_0\},F)\,,\quad x\longmapsto x(\vartheta_s(\omega))$$

is continuous. By the homotopy axiom (Axiom 1.2.5) , $K_i(\psi_\omega)=0$ so by the Product Theorem (Proposition 2.3.1 a)), $K_i(\psi')=0$.

c) follows from a), b), and Corollary 2.4.7.

d) By the commutativity of the index maps (Axiom 1.2.8), $\delta_i'=\delta_i\circ K_i(\varphi'')$ so by the Product Theorem (Proposition 2.3.1 a)),

$$\delta_i(0,(a_\omega)_{\omega\in\Gamma'})=\delta_i'(a_\omega)_{\omega\in\Gamma'}$$

for all $(a_\omega)_{\omega\in\Gamma'}\in K_i(F)^{\Gamma'}$. For $a\in K_i(F)$, by c) and by the above,

$$0=\delta_iK_i(\psi)a=\delta_i(a)_{\omega\in\Gamma}=\delta_i(a,(a)_{\omega\in\Gamma'})=$$

$$=\delta_i(a,0)+\delta_i(0,(a)_{\omega\in\Gamma'})=\delta_i(a,0)+\delta_i'(a)_{\omega\in\Gamma'}\,,$$

$\delta_i(a,0)=-\delta_i'(a)_{\omega\in\Gamma'}$. It follows for all $(a_\omega)_{\omega\in\Gamma}$,

$$\delta_i(a_\omega)_{\omega\in\Gamma}=\delta_i(a_{\omega_0},0)+\delta_i(0,(a_\omega)_{\omega\in\Gamma'})=$$

$$=-\delta_i'(a_{\omega_0})_{\omega\in\Gamma'}+\delta_i'(a_\omega)_{\omega\in\Gamma'}=\delta_i'(a_\omega-a_{\omega_0})_{\omega\in\Gamma}\,.$$

e_1) and e_2) follow from the Topological six-term sequence (Proposition 2.1.8) a_1) and b_1), respectively.

e_3) follows from the Product Theorem (Proposition 2.3.1 a)).

e_4) follows from d). ■

EXAMPLE 2.4.9 *Let $n \in \mathbb{N}$. We use the notation of* Proposition 2.4.8 *and put*

$$\Omega := \left\{ re^{\frac{2\pi ij}{n}} \;\middle|\; r \in [0,1],\, j \in \mathbb{N}_n \right\}, \quad \Gamma := \left\{ e^{\frac{2\pi ij}{n}} \;\middle|\; j \in \mathbb{N}_n \right\}, \quad \omega_0 := 1\,.$$

a) $\Omega \setminus \{\omega_0\}$ *is null-homotopic and so K-null.*

b) $K_i(\psi_{\omega_0}) : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(F)$ *is a group isomorphism.*

c) $\delta'_i : K_i(\mathscr{C}(\Gamma', F)) \approx K_i(F)^{\Gamma'} \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F))$ *is a group isomorphism.*

d) If we identify $K_i(\mathscr{C}(\Gamma', F))$ *with* $K_i(F)^{\Gamma'}$ *and* $K_i(\mathscr{C}(\Gamma, F))$ *with* $K_i(F)^{\Gamma}$ (using e.g. Lemma 2.1.4 c)) *then for all* $(a_\omega)_{\omega \in \Gamma'}$

$$K_i(\varphi'')(a_\omega)_{\omega \in \Gamma'} = (a_\omega)_{\omega \in \Gamma}\,,$$

where $a_{\omega_0} = 0$.

e) If we identify $K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F))$ *with* $K_i(F)^{\Gamma'}$ *using* $(\delta'_i)^{-1}$ *of c) then for all* $(a_\omega)_{\omega \in \Gamma}$,

$$\delta_i(a_\omega)_{\omega \in \Gamma} = (a_\omega - a_{\omega_0})_{\omega \in \Gamma'}\,.$$

f) $\Omega \in \Upsilon, \quad p(\Omega) = 1, \quad q(\Omega) = 0, \quad \Phi_{i,\Omega,F} = K_i\left(\psi_{\omega_0}\right), \quad \Omega_\Upsilon = \mathbb{C}_\Upsilon.$

a) By Proposition 2.4.1, $\Omega \setminus \{\omega_0\}$ is null-homotopic.

b) follows from a) and the Topological six-term sequence (Proposition 2.1.8 a)).

c), d), and e) follow from Proposition 2.4.8 b), c), and d), respectively.

f) follows from a) and Proposition 2.4.1. ∎

PROPOSITION 2.4.10 *Let Ω be a locally compact spaces, $\omega \in \Omega$, Ω' a compact space, and*

$$\vartheta : \Omega' \times [0,1] \longrightarrow \Omega$$

a continuous map such that $\vartheta(\omega', 0) = \omega$ for all $\omega' \in \Omega'$. Then the map

$$\mathscr{C}_0(\Omega \setminus \{\omega\}, F) \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad x \longmapsto x \circ \vartheta(\,\cdot\,, 1)$$

is K-null

For every $s \in [0,1]$ put

$$\psi_s : \mathscr{C}_0(\Omega \setminus \{\omega\}, F) \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad x \longmapsto x \circ \vartheta(\cdot, s).$$

Then for every $x \in \mathscr{C}_0(\Omega \setminus \{\omega\}, F)$ the map

$$[0,1] \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad s \longmapsto \psi_s x$$

is continuous and $\psi_0 x = 0$, so the assertion follows from the homotopy (Axiom 1.2.5). ■

PROPOSITION 2.4.11 *Let Ω be a locally compact space, Δ a closed set of Ω, Γ a compact set of Δ, $\omega_0 \in \Gamma$ such that $\mathscr{C}_0(\Delta \setminus \{\omega_0\}, F)$ is K-null, and $\vartheta : \Gamma \times [0,1] \longrightarrow \Omega$ a continuous map such that $\vartheta(\omega, 1) = \omega$ and $\vartheta(\omega, 0) = \omega_0$ for all $\omega \in \Gamma$. Then*

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F)) \times K_{i+1}(\mathscr{C}_0(\Gamma \setminus \{\omega_0\}, F)).$$

In particular if Γ is finite

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{Card\,\Gamma - 1}.$$

We use the notation of the Topological triple (Proposition 2.1.11) and put

$$\Omega_1 := \Omega \setminus \{\omega_0\}, \qquad \Omega_2 := \Omega \setminus \Gamma, \qquad \Omega_3 := \Omega \setminus \Delta.$$

By Proposition 2.4.10, $\psi_{1,2}$ is K-null and the first assertion follows from the Topological triple (Proposition 2.1.11 b_4)). The last assertion follows from the first one and from the Product Theorem (Proposition 2.3.1 a)). ■

Chapter 3

Some Selected Locally Compact Spaces

Throughout this chapter we endow $\{0,1\}$ with a group structure by identifying it with $\mathbb{Z}_2$, F denotes an E-C*-algebra, $i \in \{0,1\}$, and $n \in \mathbb{N}$.

3.1 Balls

DEFINITION 3.1.1 *We put*

$$\mathbb{B}_n := \{\ \alpha \in \mathbb{R}^n \mid \|\alpha\| \leq 1\ \}\ .$$

THEOREM 3.1.2 *Let Γ be a closed set of $\mathbb{B}_n$, $\omega_0 \in \Gamma$, and $\Gamma' := \Gamma \setminus \{\omega_0\}$.*

a) *$\mathbb{B}_n \setminus \{\omega_0\}$ is null-homotopic and so Υ-null, $\mathbb{B}_n \in \Upsilon_1$, and every exact sequence in $\mathfrak{M}_E$ belongs to $(\mathbb{B}_n)_\Upsilon$. We use in the sequel the notation of* Proposition 2.3.6 *and put there $\Omega := \mathbb{B}_n$.*

b) $K_i(\mathscr{C}_0\,(\mathbb{B}_n \setminus \Gamma, F)) \approx K_{i+1}\,(\mathscr{C}_0\,(\Gamma', F))$.

c)

$$0 \longrightarrow K_i(\mathscr{C}\,(\mathbb{B}_n, F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}\,(\Gamma, F)) \underset{\xleftarrow[\Phi_i]{}}{\xrightarrow{\delta_{1,3,i}}}$$

$$\underset{\xleftarrow[\Phi_i]{}}{\xrightarrow{\delta_{1,3,i}}} K_{i+1}(\mathscr{C}_0\,(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow 0$$

is a split exact sequence, and the maps

$$K_i(\mathscr{C}\,(\mathbb{B}_n, F)) \times K_{i+1}(\mathscr{C}_0\,(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}\,(\Gamma, F)),$$

$$(a,b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b\,,$$

$$\delta_{2,3,i} : K_i\left(\mathscr{C}_0\left(\Gamma', F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0\,(\mathbb{B}_n \setminus \Gamma, F))$$

are group isomorphisms.

d) *If $\mathbb{B}_n \setminus \Gamma \in \Upsilon$ or $\Gamma' \in \Upsilon$ then with the notation of* Corollary 2.1.9

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Gamma', F\right)\right) \longrightarrow K_{i+1}\,(\mathscr{C}_0\,(\mathbb{B}_n \setminus \{\omega_0\}, F))$$

is a group isomorphism and

$$\mathbb{B}_n \setminus \Gamma, \Gamma' \in \Upsilon\,, \qquad p(\mathbb{B}_n \setminus \Gamma) = q(\Gamma')\,, \qquad q(\mathbb{B}_n \setminus \Gamma) = p(\Gamma')\,,$$

$$\Phi_{i,(\mathbb{B}_n \setminus \Gamma),F} = \delta_{i+1} \circ \Phi_{(i+1),\Gamma',F}\ .$$

e) *Assume Γ finite.*

e_1)

$$(\delta_{2,3,i})^{-1} : K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow K_i(F)^{\Gamma'}$$

is a group isomorphism.

e_2)

$$K_i(\psi_{1,3}) : K_i(\mathscr{C}(\mathbb{B}_n, F)) \approx K_i(F) \longrightarrow K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F)^{\Gamma},$$

$$a \longmapsto (a)_{\omega\in\Gamma},$$

and, if we identify $K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F))$ with $K_i(F)^{\Gamma'}$ using the above group isomorphism $(\delta_{2,3,i})^{-1}$, then

$$\delta_{1,3,i} : K_i(\mathscr{C}(\Gamma, F)) \longrightarrow K_i(F)^{Card\,\Gamma'}, \quad (a_\omega)_{\omega\in\Gamma} \longmapsto (a_\omega - a_{\omega_0})_{\omega\in\Gamma'} .$$

e_3)

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \qquad p(\mathbb{B}_n \setminus \Gamma) = 0, \qquad q(\mathbb{B}_n \setminus \Gamma) = Card\,\Gamma',$$

$$\Phi_{i,(\mathbb{B}_n\setminus\Gamma),F} = \delta_{2,3,(i+1)} \circ \Phi_{(i+1),\Gamma',F},$$

a) Since $\mathbb{B}_n$ is homeomorphic to $[0,1]^n$, it follows from Corollary 2.4.2 that $\mathscr{C}_0(\Omega \setminus \{\omega_0\}, \mathbb{C})$ is null-homotopic and $\mathbb{B}_n \in \Upsilon_1$. By Proposition 1.5.4 d), $\mathbb{B}_n \setminus \{\omega_0\}$ is Υ-null and by Proposition 1.6.6, every exact sequence in $\mathfrak{M}_E$ belongs to $(\mathbb{B}_n)_\Upsilon$.

b), c), d), e_1), and e_3) follow from a) and Proposition 2.3.6.

e_2) follows from a) and Proposition 2.4.8 $e_3), e_4)$. ■

Remark. By b), $K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F))$ depends only on $K_{i+1}(\mathscr{C}_0(\Gamma', F))$ and not on n or on the embedding of Γ in $\mathbb{B}_n$.

COROLLARY 3.1.3 *Let $(\Gamma_j)_{j\in J}$ be a finite family of pairwise disjoint closed sets of $\mathbb{B}_n$, $J \neq \emptyset$, and for every $j \in J$ let $\omega_j \in \Gamma_j$ such that $\mathscr{C}_0(\Gamma_j \setminus \{\omega_j\}, F)$ is K-null. Then*

$$K_i\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus \bigcup_{j\in J} \Gamma_j, F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \{\, \omega_j \mid j \in J \,\}, F)) \approx K_{i+1}(F)^{Card\,J-1}$$

Put $\Gamma := \bigcup\limits_{j\in J} (\Gamma_j \setminus \{\omega_{\mathrm{j}}\})$,

$$\psi : \mathscr{C}_0 \left(\mathrm{I\!B}_n \setminus \left\{ \ \omega_j \ \middle| \ j \in J \right\}, F\right) \longrightarrow \mathscr{C}_0 (\Gamma, F) \,, \quad x \longmapsto x|\Gamma \,,$$

and denote by $\varphi : \mathscr{C}_0 \left(\mathrm{I\!B}_n \setminus \bigcup\limits_{j\in J} \Gamma_j, F \right) \longrightarrow \mathscr{C}_0 \left(\mathrm{I\!B}_n \setminus \left\{ \ \omega_j \ \middle| \ j \in J \right\}, F\right)$ the inclusion map. Then

$$0 \longrightarrow \mathscr{C}_0 \left(\mathrm{I\!B}_n \setminus \bigcup_{j\in J} \Gamma_j, F \right) \xrightarrow{\varphi} \mathscr{C}_0 \left(\mathrm{I\!B}_n \setminus \left\{ \ \omega_j \ \middle| \ j \in J \right\}, F\right) \xrightarrow{\psi} \mathscr{C}_0 (\Gamma, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. By the Product Theorem (Proposition 2.3.1 c)), $\mathscr{C}_0 (\Gamma, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 b)) and Theorem 3.1.2 e_1),

$$K_i \left(\mathscr{C}_0 \left(\mathrm{I\!B}_n \setminus \bigcup_{j\in J} \Gamma_j, F \right) \right) \approx$$

$$\approx K_i(\mathscr{C}_0 \left(\mathrm{I\!B}_n \setminus \left\{ \ \omega_j \ \middle| \ j \in J \right\}, F\right)) \approx K_{i+1}(F)^{Card\, J-1} \,.$$

■

COROLLARY 3.1.4 *Let $(k_j)_{j\in J}$ be a finite family in $\mathrm{I\!N}$ and for every $j \in J$ let Γ_j be a nonempty finite subset of $\mathrm{I\!B}_{k_j}$. If Ω denotes the Alexandroff compactification of the topological sum of the family $(\mathrm{I\!B}_{k_j} \setminus \Gamma_j)_{j\in J}$ then*

$$\Omega \in \Upsilon, \qquad p(\Omega) = 1 \,, \qquad q(\Omega) = \sum_{j\in J} (Card\, \Gamma_j - 1) \,.$$

For every $j \in J$ let $\omega_j \in \Gamma_j$. By Theorem 3.1.2 a), $\mathrm{I\!B}_{k_j} \setminus \{\omega_{\mathrm{j}}\}$ is Υ-null and the assertion follows from Corollary 2.3.5 b). ■

COROLLARY 3.1.5 *If Ω is a path connected compact space, $\omega \in \Omega$, and $\omega' \in \mathrm{I\!B}_n \times \Omega$ then*

$$K_i (\mathscr{C}_0 (\Omega \setminus \{\omega\}, F)) \approx K_i \left(\mathscr{C}_0 \left(\mathrm{I\!B}_n \times \Omega \setminus \{\omega'\}, F\right)\right) \,.$$

B¡ Theorem 3.1.2 a), $\mathscr{C}_0 (\mathrm{I\!B}_n \setminus \{\omega_0\}, F)$ is K-null for every $\omega_0 \in \mathrm{I\!B}_n$ and the assertion follows from Corollary 2.4.5. ■

COROLLARY 3.1.6 *Let Γ be a closed set of $\mathbb{B}_n$ and Ω an open set of $\mathbb{B}_n$, $\Omega \subset \Gamma$. Then for all $\omega \in \Gamma \setminus \Omega$,*

$$K_i(\mathscr{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_i(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) ,$$

$$K_i(\mathscr{C}_0(\Gamma \setminus \Omega, F)) \approx K_i(\mathscr{C}(\Gamma, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) .$$

By Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) ,$$

$$K_i(\mathscr{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus (\Gamma \setminus \Omega), F))$$

and by the Product Theorem (Proposition 2.3.1a)),

$$K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus (\Gamma \setminus \Omega), F)) \approx K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) ,$$

so

$$K_i(\mathscr{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_i(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) .$$

The last relation follows from the Alexandroff K-theorem (Proposition 2.2.1 a)). ■

COROLLARY 3.1.7 *If Ω is an open set of $\mathbb{B}_n$, $\Omega \neq \mathbb{B}_n$, and Γ a compact set of Ω then*

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega, F)) \times K_{i+1}(\mathscr{C}(\Gamma, F)) .$$

Let $\omega \in \mathbb{B}_n \setminus \Omega$. By Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(\mathscr{C}_0((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}, F)) ,$$

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}_0(((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}) \cup \Gamma, F)) .$$

By the Product Theorem (Proposition 2.3.1 a)),

$$K_{i+1}(\mathscr{C}_0(((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}) \cup \Gamma, F)) \approx$$

$$\approx K_{i+1}(\mathscr{C}_0((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}(\Gamma, F)) ,$$

so

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega, F)) \times K_{i+1}(\mathscr{C}(\Gamma, F)) .$$ ■

3.2 Euclidean Spaces and Spheres

DEFINITION 3.2.1 *We put*

$$\mathbb{S}_{n-1} := \{ \alpha \in \mathbb{R}^n \mid \|\alpha\| = 1 \} , \qquad \mathbb{T} := \mathbb{S}_1 .$$

THEOREM 3.2.2

a) $\mathbb{R}^n \in \Upsilon, \qquad p(\mathbb{R}^n) = \frac{1+(-1)^n}{2}, \qquad q(\mathbb{R}^n) = \frac{1-(-1)^n}{2},$

$$\mathbb{R}_\Upsilon \subset (\mathbb{R}^n)_\Upsilon , \qquad K_i(\mathscr{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F) .$$

b) $\mathbb{S}_n \in \Upsilon, \quad p(\mathbb{S}_n) = \frac{3+(-1)^n}{2}, \quad q(\mathbb{S}_n) = \frac{1-(-1)^n}{2}, \quad \mathbb{R}_\Upsilon \subset (\mathbb{S}_n)_\Upsilon,$

$$K_i(\mathscr{C}(\mathbb{S}_n, F)) \approx$$

$$\approx \begin{cases} K_i(F)^2 & \text{if} \quad n \quad \text{is even} \\ K_i(F) \times K_{i+1}(F) & \text{if} \quad n \quad \text{is odd} \end{cases} = K_i(F) \times K_{i+n}(F) ,$$

and the map

$$K_i(F) \times K_{i+n}(F) \longrightarrow K_i(\mathscr{C}(\mathbb{S}_n, F)), \quad (a,b) \longmapsto K_i(\lambda)a + K_{i+n}(\varphi)b$$

is a group isomorphism, where $\varphi : \mathscr{C}_0(\mathbb{R}^n, F) \approx K_{i+n}(F) \longrightarrow \mathscr{C}(\mathbb{S}_n, F)$ *denotes the inclusion map and*

$$\lambda : F \longrightarrow \mathscr{C}(\mathbb{S}_n, F) , \quad x \longmapsto x1_{\mathscr{C}(\mathbb{S}_n, \mathbb{C})} .$$

c) Let Γ *be a closed set of* $\mathbb{R}^n$, $\Gamma \neq \mathbb{R}^n$.

c$_1$*) The map*

$$\mathscr{C}_0(\mathbb{R}^n, F) \longrightarrow \mathscr{C}_0(\Gamma, F) , \quad x \longmapsto x|\Gamma$$

is K-null.

c$_2$*) If* Γ *is compact then*

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathscr{C}(\Gamma, F)) .$$

If in addition $\Gamma\in\Upsilon$ *then* $\mathbb{R}^n\setminus\Gamma\in\Upsilon$, *and*

$$p(\mathbb{R}^n\setminus\Gamma)=\begin{cases} q(\Gamma)+1 & \text{if } n \text{ is even}\\ q(\Gamma) & \text{if } n \text{ is odd}\end{cases},$$

$$q(\mathbb{R}^n\setminus\Gamma)=\begin{cases} p(\Gamma) & \text{if } n \text{ is even}\\ p(\Gamma)+1 & \text{if } n \text{ is odd}\end{cases}.$$

d) If Γ *is finite then* $\mathbb{R}^n\setminus\Gamma\in\Upsilon$, *and*

$$p(\mathbb{R}^n\setminus\Gamma)=\begin{cases} 1 & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases},$$

$$q(\mathbb{R}^n\setminus\Gamma)=\begin{cases} \operatorname{Card}\Gamma & \text{if } n \text{ is even}\\ \operatorname{Card}\Gamma+1 & \text{if } n \text{ is odd}\end{cases}.$$

e) Let Γ *be a closed set of* $\mathbf{S}_n$, $\Gamma\neq\mathbf{S}_n$, $\omega\in\Gamma$, *and* $\Gamma':=\Gamma\setminus\{\omega\}$.

e_1) $K_i(\mathscr{C}_0(\mathbf{S}_n\setminus\Gamma,F))\approx K_{i+n}(F)\times K_{i+1}(\mathscr{C}_0(\Gamma\setminus\{\omega\},F))$.

e_2) If $\Gamma'\in\Upsilon$ *then* $\mathbf{S}_n\setminus\Gamma\in\Upsilon$, *and*

$$p(\mathbf{S}_n\setminus\Gamma)=\begin{cases} q(\Gamma')+1 & \text{if } n \text{ is even}\\ q(\Gamma') & \text{if } n \text{ is odd}\end{cases},$$

$$q(\mathbf{S}_n\setminus\Gamma)=\begin{cases} p(\Gamma') & \text{if } n \text{ is even}\\ p(\Gamma')+1 & \text{if } n \text{ is odd}\end{cases}.$$

e_3) If Γ *is finite, then* $\mathbf{S}_n\setminus\Gamma\in\Upsilon$, *and*

$$p(\mathbf{S}_n\setminus\Gamma)=\begin{cases} 1 & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases},$$

$$q(\mathbf{S}_n\setminus\Gamma)=\begin{cases} \operatorname{Card}\Gamma' & \text{if } n \text{ is even}\\ \operatorname{Card}\Gamma & \text{if } n \text{ is odd}\end{cases}.$$

f) If $m\in\mathbb{N}, m<n$, *then*

$$K_i(\mathscr{C}_0(\mathbf{S}_n\setminus\mathbf{S}_m,F))\approx K_i(\mathscr{C}_0(\mathbb{R}^n\setminus\mathbb{R}^m,F))\approx K_i(F)\times K_{i+n-m+1}(F)\,.$$

g) For $m\in\mathbb{N}, m<n$,

$$K_i(\mathscr{C}_0(\mathbb{B}_n\setminus\mathbf{S}_m,F))\approx K_{i+m+1}(F)\,.$$

a) Since $\mathbb{R}$ is homeomorphic to $]0,1[=\mathbb{B}_1\setminus\{-1,1\}$ we get

$$\mathbb{R}\in\Upsilon\,,\qquad p(\mathbb{R})=0\,,\qquad q(\mathbb{R})=1$$

from Theorem 3.1.2 e_3) and the assertion follows from Corollary 1.5.12.

b) Since $\boldsymbol{S}_n$ is homeomorphic to the Alexandroff compactification of $\mathbb{R}^n$, b) follows from a) and the Alexandroff K-theorem (Theorem 2.2.1 a),b)).

c1) We may assume $0\in\mathbb{R}^n\setminus\Gamma$. Put

$$\vartheta:\Gamma\times]0,1]\longrightarrow\mathbb{R}^n\,,\quad(\omega,s)\longmapsto\frac{1}{s}\,\omega$$

and for every $s\in[0,1]$

$$\psi_s:\mathscr{C}_0\left(\mathbb{R}^n,F\right)\longrightarrow\mathscr{C}_0\left(\Gamma,F\right)\,,\quad x\longmapsto\begin{cases}x\circ\vartheta(\cdot\,,s) & \textit{if}\quad s\neq0\\ 0 & \textit{if}\quad s=0\end{cases}\,.$$

Then for every $x\in\mathscr{C}_0\left(\mathbb{R}^n,F\right)$,

$$[0,1]\longrightarrow\mathscr{C}_0\left(\Gamma,F\right)\,,\quad s\longmapsto\psi_s x$$

is continuous, $\psi_1 x=x|\Gamma$, and $\psi_0 x=0$. Thus the assertion follows from the homotopy axiom (Axiom 1.2.5).

c2) We identify the homeomorphic spaces $\{\,\alpha\in\mathbb{R}^n\mid\|\alpha\|<1\,\}$ and $\mathbb{R}^n$, put $\omega:=(1,0,\cdots,0)\in\mathbb{B}_n$ and

$$\psi:\mathscr{C}_0\left(\mathbb{B}_n\setminus\{\omega\},F\right)\longrightarrow\mathscr{C}_0\left((\mathbf{S}_{n-1}\setminus\{\omega\})\cup\Gamma,F\right)\,,\quad x\longmapsto x|((\mathbf{S}_{n-1}\setminus\{\omega\})\cup\Gamma)\,,$$

and denote by $\varphi:\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma,F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{B}_n\setminus\{\omega\},F\right)$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma,F\right)\xrightarrow{\varphi}\mathscr{C}_0\left(\mathbb{B}_n\setminus\{\omega\},F\right)\xrightarrow{\psi}\mathscr{C}_0\left((\mathbf{S}_{n-1}\setminus\{\omega\})\cup\Gamma,F\right)\longrightarrow0\,.$$

By Theorem 3.1.2 a), $\mathscr{C}_0\left(\mathbb{B}_n\setminus\{\omega\},F\right)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)), the map

$$\delta_{i+1}:K_{i+1}(\mathscr{C}_0\left((\mathbf{S}_{n-1}\setminus\{\omega\})\cup\Gamma,F\right))\longrightarrow K_i(\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma,F\right))$$

is a group isomorphism. By the Product Theorem (Proposition 2.3.1 a),b)),

$$K_{i+1}(\mathscr{C}_0\left((\mathbf{S}_{n-1}\setminus\{\omega\})\cup\Gamma,F\right))\approx K_{i+1}(\mathscr{C}_0\left(\mathbf{S}_{n-1}\setminus\{\omega\},F\right))\times K_{i+1}(\mathscr{C}\left(\Gamma,F\right))\,,$$

and $\Gamma \in \Upsilon$ implies $\mathbb{R}^n \setminus \Gamma \in \Upsilon$. By a), $K_{i+1}(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \{\omega\}, F)) \approx K_{i+n}(F)$ so

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathscr{C}(\Gamma, F))$$

as well as the last assertions.

d) follows from c) and the Product Theorem (Proposition 2.3.1 a),b)).

e) $\mathbf{S}_n \setminus \Gamma$ is homeomorphic to $\mathbb{R}^n \setminus (\Gamma \setminus \{\omega\})$ and the assertion follows from c) and d).

f) Step 1

$$K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F))$$

Let $\omega \in \mathbf{S}_m$. Then $\mathbf{S}_n \setminus \mathbf{S}_m = (\mathbf{S}_n \setminus \{\omega\}) \setminus (\mathbf{S}_m \setminus \{\omega\})$. Since $(\mathbf{S}_n \setminus \{\omega\}) \setminus (\mathbf{S}_m \setminus \{\omega\})$ is homeomorphic to $\mathbb{R}^n \setminus \mathbb{R}^m$ we get

$$K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) .$$

Step 2

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_i(F) \times K_{i+n-m+1}(F)$$

We identify $\mathbb{R}^n \setminus \mathbb{R}^m$ with $\left\{ \alpha \in \mathbb{B}_n \mid \|\alpha\| < 1 , \sum_{j=m+1}^{n} \alpha_j^2 \neq 0 \right\}$, put

$$\psi : \mathbb{B}_n \setminus \mathbb{B}_m \longrightarrow \mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1} , \quad x \longmapsto x|(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}) ,$$

and denote by $\varphi : \mathbb{R}^n \setminus \mathbb{R}^m \longrightarrow \mathbb{B}_n \setminus \mathbb{B}_m$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F) \xrightarrow{\varphi} \mathscr{C}_0(\mathbb{B}_n \setminus \mathbb{B}_m, F) \xrightarrow{\psi} \mathscr{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}, F) \longrightarrow 0 .$$

By Proposition 2.4.1, $\mathscr{C}_0(\mathbb{B}_n \setminus \mathbb{B}_m, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)) and Step 1,

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}, F)) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{R}^{n-1} \setminus \mathbb{R}^{m-1}, F\right)\right) .$$

For $m = 1$, by e_1),

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_0, F)) \approx K_{i+n}(F) \times K_i(F) .$$

By induction and by the above,

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_{i+n-m+1}\left(\mathscr{C}_0\left(\mathbb{R}^{n-m+1} \setminus \mathbb{R}, F\right)\right) \approx$$

$$\approx K_{i+n-m+1}(F) \times K_i(F) .$$

g) Let $\omega \in \mathbf{S}_m$. Since $\mathbf{S}_m \setminus \{\omega\}$ is homeomorph to $\mathbb{R}^m$, by a),

$$K_i(\mathscr{C}_0(\mathbf{S}_m \setminus \{\omega\}, F)) \approx K_{i+m}(F) .$$

By Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_m, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbf{S}_m \setminus \{\omega\}, F)) \approx K_{i+1+m}(F) . \qquad \blacksquare$$

EXAMPLE 3.2.3 *Put*

$$\Omega_1 := \mathbf{S}_1 \cup \left\{ re^{\frac{2\pi i j}{n}} \;\middle|\; r \in [0,1],\ j \in \mathbb{N}_n \right\},$$

$$\Omega_2 := \mathbf{S}_2 \cup \{ \alpha \in \mathbb{B}_3 \mid \alpha_3 = 0 \} \cup \{ \alpha \in \mathbb{B}_3 \mid \alpha_1 = \alpha_2 = 0 \} ,$$

$$\Omega_3 := \mathbf{S}_{n-1} \cup \left(\bigcup_{j \in \mathbb{N}_n} \{ \alpha \in \mathbb{B}_n \mid \alpha_j = 0 \} \right) .$$

a) $K_i(\mathscr{C}(\Omega_1, F)) = K_i(F) \times K_{i+1}(F)^n$.

b) $K_i(\mathscr{C}(\Omega_2, F)) \approx K_i(F)^3 \times K_{i+1}(F)^2$.

c) $K_i(\mathscr{C}(\Omega_3, F)) = K_i(F) \times K_{i+n+1}(F)^{2^n}$.

a) By Theorem 3.2.2 b) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega_1, F)) \approx K_i(F)^n$$

and by Theorem 3.1.2 a),b),c),

$$K_i(\mathscr{C}(\Omega_1, F)) \approx K_i(\mathscr{C}(\mathbb{B}_2, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega_1, F)) \approx K_i(F) \times K_{i+1}(F)^n .$$

b) By Theorem 3.2.2 a),b),

$$\mathbb{R}^2, \mathbf{S}_1 \in \Upsilon, \qquad p(\mathbb{R}^2) = 1, \qquad q(\mathbb{R}^2) = 0, \qquad p(\mathbf{S}_1) = 1, \qquad q(\mathbf{S}_1) = 1,$$

so by Corollary 1.5.11 d_1),

$$K_i\left(\mathscr{C}_0\left(\mathrm{I\!R}^2\times\mathbb{S}_1,F\right)\right)\approx K_i(F)\times K_{i+1}(F)\ .$$

Since $\mathrm{I\!B}_3\setminus\Omega_2$ is homeomorphic to the topological sum of two copies of $\mathrm{I\!R}^2\times\mathbb{S}_1$ we get by the Product Theorem (Proposition 2.3.1 a))

$$K_i\left(\mathscr{C}_0\left(\mathrm{I\!B}_3\setminus\Omega_2,F\right)\right)\approx K_i(F)^2\times K_{i+1}(F)^2\ .$$

By Theorem 3.1.2 a),b),c),

$$K_i\left(\mathscr{C}\left(\Omega_2,F\right)\right)\approx K_i\left(\mathscr{C}\left(\mathrm{I\!B}_3,F\right)\right)\times K_{i+1}\left(\mathscr{C}_0\left(\mathrm{I\!B}_3\setminus\Omega_2,F\right)\right)\approx K_i(F)^3\times K_{i+1}(F)^2.$$

c) By Theorem 3.2.2 a), $K_i\left(\mathscr{C}_0\left(\mathrm{I\!R}^n,F\right)\right)\approx K_{i+n}(F)$. Since $\mathrm{I\!B}_n\setminus\Omega_3$ is homeomorphic to the topological sum of 2^n copies of $\mathrm{I\!R}^n$, we get by the Product Theorem (Proposition 2.3.1 a)) $K_i\left(\mathscr{C}_0\left(\mathrm{I\!B}_n\setminus\Omega_3,F\right)\right)\approx K_{i+n}(F)^{2^n}$. By Theorem 3.1.2 a),b),c),

$$K_i\left(\mathscr{C}\left(\Omega_3,F\right)\right)\approx K_i\left(\mathscr{C}\left(\mathrm{I\!B}_n,F\right)\right)\times K_{i+1}\left(\mathscr{C}_0\left(\mathrm{I\!B}_n\setminus\Omega_3,F\right)\right)\approx$$

$$\approx K_i(F)\times K_{i+n+1}(F)^{2^n}\ . \qquad \blacksquare$$

Remark. The above a) and b) will be generalized in Example 3.5.11 b) and c), respectively.

COROLLARY 3.2.4 *Let $(k_j)_{j\in J}$ be a finite family in $\mathrm{I\!N}$ and*

$$p:=Card\left\{\,j\in J\mid k_j\text{ is even}\right\},\qquad q:=Card\left\{\,j\in J\mid k_j\text{ is odd}\right\}\ .$$

a) If Ω denotes the Alexandroff compactification of the topological sum of the family $(\mathrm{I\!R}^{k_j})_{j\in J}$ then

$$\Omega\in\Upsilon,\qquad \mathrm{I\!R}_\Upsilon\subset\Omega_\Upsilon,\qquad p(\Omega)=p+1,\qquad q(\Omega)=q\ .$$

b) For every $j\in J$ let $\omega_j\in\mathbb{S}_{k_j}$ and let Ω' denote the compact space obtained from the topological sum of the family $(\mathbb{S}_{k_j})_{j\in J}$ by identifying all the points of the family $(\omega_j)_{j\in J}$. If $J\neq\emptyset$ then

$$\Omega'\in\Upsilon,\qquad \mathrm{I\!R}_\Upsilon\subset\Omega'_\Upsilon,\qquad p(\Omega')=p+1,\qquad q(\Omega')=q\ .$$

In particular if $k_j=1$ for all $j\in J$ then $p(\Omega')=1$, $q(\Omega')=Card\,J$.

a) By Theorem 3.2.2 a), $\mathbb{R}^{k_j} \in \Upsilon$, $\mathbb{R}_\Upsilon \subset (\mathbb{R}^{k_j})_\Upsilon$,

$$p\left(\mathbb{R}^{k_j}\right) = \begin{cases} 1 & \text{if} \quad k_j \text{ is even} \\ 0 & \text{if} \quad k_j \text{ is odd} \end{cases}, \qquad q\left(\mathbb{R}^{k_j}\right) = \begin{cases} 0 & \text{if} \quad k_j \text{ is even} \\ 1 & \text{if} \quad k_j \text{ is odd} \end{cases}$$

for every $j \in J$. The assertion follows now from the Product Theorem (Proposition 2.3.1 b)) and from Alexandroff's K-theorem (Proposition 2.2.1 b)).

b) follows from a) since Ω and Ω' are homeomorphic. ∎

COROLLARY 3.2.5 *Let $(k_j)_{j\in J}$ be a finite family in $\mathbb{N}$,*

$$p := Card\left\{\, j \in J \mid k_j \text{ is even} \right\}, \qquad q := Card\left\{\, j \in J \mid k_j \text{ is odd} \right\},$$

$(\Gamma_j)_{j\in J}$ a pairwise disjoint family of closed sets of $\mathbb{B}_n$ such that Γ_j is homeomorphic to $\mathbf{S}_{k_j}$ for every $j \in J$, and $\Gamma := \bigcup_{j\in J} \Gamma_j$. Then

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset (\mathbb{B}_n \setminus \Gamma)_\Upsilon, \qquad p(\mathbb{B}_n \setminus \Gamma) = q, \qquad q(\mathbb{B}_n \setminus \Gamma) = 2p - 1 .$$

By Theorem 3.2.2 a),b), for $j \in J$,

$$\mathbb{R}^{k_j}, \mathbf{S}_{k_j} \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset (\mathbb{R}^{k_j})_\Upsilon \cap (\mathbf{S}_{k_j})_\Upsilon,$$

$$p\left(\mathbb{R}^{k_j}\right) = \frac{1+(-1)^{k_j}}{2}, \qquad q\left(\mathbb{R}^{k_j}\right) = \frac{1-(-1)^{k_j}}{2},$$

$$p\left(\mathbf{S}_{k_j}\right) = \frac{3+(-1)^{k_j}}{2}, \qquad q\left(\mathbf{S}_{k_j}\right) = \frac{1-(-1)^{k_j}}{2}.$$

Let $\omega \in \Gamma$ and $\Gamma' := \Gamma \setminus \{\omega\}$. By the Product Theorem (Proposition 2.3.1 b)),

$$\Gamma' \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset \Gamma'_\Upsilon, \qquad p(\Gamma') = 2p - 1, \qquad q(\Gamma') = q,$$

so by Theorem 3.1.2 d),

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset (\mathbb{B}_n \setminus \Gamma)_\Upsilon, \qquad p(\mathbb{B}_n \setminus \Gamma) = q, \qquad q(\mathbb{B}_n \setminus \Gamma) = 2p - 1 . \quad ∎$$

COROLLARY 3.2.6 *If Ω is a connected closed set of $\mathbb{B}_2$ possessing a triangulation with r_0 vertices, r_1 chords, and r_2 triangles then*

$$K_i\left(\mathscr{C}\left(\Omega, F\right)\right) \approx K_i\left(F\right) \times K_{i+1}\left(F\right)^{1-r_0+r_1-r_2} .$$

Sketch of a proof. If Ω has k holes then $r_0 - r_1 + r_2 + k = 1$. By Theorem 3.1.2 c),

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega, F)) \ .$$

By Theorem 3.2.2 a) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega, F)) \approx K_i(F)^k$$

so

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{1-r_0+r_1-r_2} \ . \qquad \blacksquare$$

COROLLARY 3.2.7 *We identify the homeomorphic spaces* $\mathbb{R}^n$ *and*

$$\{\, \alpha \in \mathbb{R}^n \mid \|\alpha\| < 1 \,\} \ .$$

Let Γ *be a finite subset of* $\mathbb{R}^n$, Δ *a subset of* Γ, $\omega \in \Delta$, $\Gamma' := \Gamma \setminus \{\omega\}$, $\Delta' := \Delta \setminus \{\omega\}$. *We use the notation of the* Topological triple (Proposition 2.1.11) *and put*

$$\Omega_1 := \mathbb{B}_n \setminus \{\omega\}, \qquad \Omega_2 := \mathbb{R}^n \setminus \Delta, \qquad \Omega_3 := \mathbb{R}^n \setminus \Gamma .$$

a) $\delta_{1,2,i}$ *and* $\delta_{1,3,i}$ *are group isomorphisms.*

b) $\psi_{2,3}$ *is K-null.*

c) If we put $\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi') \circ (\delta_{1,2,(i+1)})^{-1}$ *then*

$$0 \longrightarrow K_{i+1}(\mathscr{C}(\Gamma \setminus \Delta, F)) \xrightarrow{\delta_{2,3,(i+1)}} K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \underset{\Phi_i}{\overset{K_i(\varphi_{2,3})}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{K_i(\varphi_{2,3})}{\rightleftarrows}} K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Delta, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}(\mathscr{C}(\Gamma \setminus \Delta, F)) \times K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Delta, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)),$$

$$(a, b) \longmapsto \delta_{2,3,(i+1)} a + \Phi_i b$$

is a group isomorphism.

By Theorem 3.1.2 a), $\mathscr{C}_0(\Omega_1, F)$ is K-null and by Proposition 2.4.10, $\psi_{2,3}$ is K-null. By the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\psi \circ \varphi') = id_{K_i(\mathscr{C}_0(\Omega_1 \setminus \Omega_2, F))}$$

and a) and c) follow from the Topological triple (Proposition 2.1.11 c)). ■

COROLLARY 3.2.8 *Let $\omega \in \mathbf{S}_{n-1}$. We use the notation of the* Topological triple (Proposition 2.1.11) *and put*

$$\Omega_1 := \mathbb{B}_n\,, \qquad \Omega_2 := \mathbb{B}_n \setminus \{\omega\}\,, \qquad \Omega_3 := \mathbb{B}_n \setminus \mathbf{S}_{n-1}\,.$$

a) $\varphi_{1,3}$ *is K-null.*

b) $\delta_{2,3,i} \ : \ K_i(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \{\omega\}, F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$ *is a group isomorphism.*

c) If we put $\Phi_i := K_i(\varphi) \circ (\delta_{2,3,i})^{-1}$ *then*

$$0 \longrightarrow K_i(\mathscr{C}(\mathbb{B}_n, F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}(\mathbf{S}_{n-1}, F)) \underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}} K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(\mathscr{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1}, F)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

is a group isomorphism.

d) Let $\phi : G \longrightarrow H$ *be a morphism in* $\mathfrak{M}_E$ *and put*

$$\phi_{\mathbb{B}} : \mathscr{C}(\mathbb{B}_n, G) \longrightarrow \mathscr{C}(\mathbb{B}_n, H)\,, \quad x \longmapsto \phi \circ x\,,$$

$$\phi_{\mathbb{S}} : \mathscr{C}(\mathbf{S}_{n-1}, G) \longrightarrow \mathscr{C}(\mathbf{S}_{n-1}, H)\,, \quad x \longmapsto \phi \circ x\,,$$

$$\phi_{\mathbb{B},\mathbb{S}} : \mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, G) \longrightarrow \mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, H)\,, \quad x \longmapsto \phi \circ x\,.$$

If we identify $K_i(\mathscr{C}(\mathbf{S}_{n-1}, F))$ *with*

$$K_i(\mathscr{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$$

for $F \in \{\mathrm{G},\mathrm{H}\}$ *using the isomorphism of c) then*

$$K_i(\phi_{\mathbb{S}}) : K_i(\mathscr{C}(\mathbf{S}_{n-1},G)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1},H)),$$

$$(a,b) \longmapsto (K_i(\phi_{\mathbb{B}})a, K_{i+1}(\phi_{\mathbb{B},\mathbb{S}})b)\ .$$

By Theorem 3.1.2 a), $\mathscr{C}_0(\mathbb{B}_n \setminus \{\omega\},F)$ is K-null and the assertion follows from the Topological triple (Proposition 2.1.11 a)) and Corollary 2.1.12 b). ■

PROPOSITION 3.2.9 *Put*

$$\Omega := \mathbb{B}_{n+1} \setminus \{\ \alpha \in \mathbf{S}_n \mid \alpha_{n+1} = 0\}\ ,$$

$$\Omega' := \mathbf{S}_n \setminus \{\ \alpha \in \mathbf{S}_n \mid \alpha_{n+1} = 0\}\ ,$$

$$\psi : \mathscr{C}_0(\Omega,F) \longrightarrow \mathscr{C}_0\left(\Omega',F\right)\ , \quad x \longmapsto x|\Omega'$$

and denote by

$$\varphi : \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n,F) \longrightarrow \mathscr{C}_0(\Omega,F)$$

the inclusion map and by δ_i *the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n,F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega,F) \xrightarrow{\psi} \mathscr{C}_0\left(\Omega',F\right) \longrightarrow 0\ .$$

a)

$$K_i(\mathscr{C}_0(\Omega,F)) \approx K_{i+n}(F)\ , \qquad K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \approx K_{i+n}(F)^2\ ,$$

$$K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n,F)) \approx K_{i+n}(F)\ .$$

b) If we identify the groups of a) then

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n,F))\ , \quad (a,b) \longmapsto a+b\ ,$$

$$0 \longrightarrow K_i(\mathscr{C}_0(\Omega,\cdot)) \xrightarrow{K_i(\psi)} K_i\left(\mathscr{C}_0\left(\Omega',\cdot\right)\right) \xrightarrow{\delta_i} K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n,\cdot)) \longrightarrow 0$$

is an exact sequence, and there is a group automorphism $\Phi_i : K_{i+n}(F) \longrightarrow K_{i+n}(F)$ *such that*

$$K_i(\psi) : K_i(\mathscr{C}_0(\Omega,F)) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right)\ , \quad a \longmapsto (\Phi_i a, -\Phi_i a)\ .$$

c) If

$$\lambda' : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\mathbb{B}_{n+1}, F)\,,$$

$$\lambda'' : \mathscr{C}_0(\Omega', F) \longrightarrow \mathscr{C}(\mathbf{S}_n, F)$$

denote the inclusion maps and if we identify $K_i(\mathscr{C}_0(\Omega', F))$ with $K_{i+n}(F)^2$ using a) and $K_i(\mathscr{C}(\mathbf{S}_n, F))$ with $K_i(F) \times K_{i+n}(F)$ using Theorem 3.2.2 b) *then λ' is K-null and*

$$K_i(\lambda'') : K_i(\mathscr{C}(\Omega', F)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_n, F))\,, \quad (a,b) \longmapsto (0, a+b)\,.$$

a) By Theorem 3.2.2 a), $K_i(\mathscr{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F)$. Since $\mathbb{B}_{n+1} \setminus \mathbf{S}_n$ is homeomorphic to $\mathbb{R}^{n+1}$, $K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) \approx K_{i+n}(F)$. Since Ω' is homeomorphic to the topological sum of $\mathbb{R}^n$ and $\mathbb{R}^n$, $K_i(\mathscr{C}_0(\Omega', F)) \approx K_{i+n}(F)^2$ by the Product Theorem (Proposition 2.3.1 a)). Put

$$\Gamma := \{\, \alpha \in \Omega \mid \alpha_{n+1} = 0 \}$$

and for every $s \in]0,1]$

$$\vartheta_s : \Omega \setminus \Gamma \longrightarrow \Omega \setminus \Gamma\,, \quad (\alpha_j)_{j \in \mathbb{N}_{n+1}} \longmapsto ((\alpha_j)_{j \in \mathbb{N}_n}, s\alpha_{n+1})\,.$$

By Proposition 2.4.1, $\mathscr{C}_0(\Omega \setminus \Gamma, F)$ is K-null, so by the Topological six-term sequence (Proposition 2.1.8 a)), $K_i(\mathscr{C}_0(\Omega, F)) \approx K_i(\mathscr{C}_0(\Gamma, F))$. Since Γ is homeomorphic to $\mathbb{R}^n$, $K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+n}(F)$ by the above.

b) Put $\omega := (1, 0, \cdots, 0) \in \mathbb{B}_{n+1}$,

$$\psi' : \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F) \longrightarrow \mathscr{C}_0(\mathbf{S}_n \setminus \{\omega\}, F)\,, \quad x \longmapsto x|(\mathbf{S}_n \setminus \{\omega\})\,,$$

and denote by

$$\varphi' : \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F)\,,$$

$$\varphi'' : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F)$$

$$\varphi''' : \mathscr{C}_0(\Omega', F) \longrightarrow \mathscr{C}_0(\mathbf{S}_n \setminus \{\omega\}, F)$$

the inclusion maps and by δ_i' the six-term sequence index maps associated with the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \xrightarrow{\varphi'} \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F) \xrightarrow{\psi'} \mathscr{C}_0(\mathbf{S}_n \setminus \{\omega\}, F) \longrightarrow 0\,.$$

By Theorem 3.1.2 a), $\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\{\omega\},F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)),

$$\delta_i' : K_i(\mathscr{C}_0(\mathbf{S}_n\setminus\{\omega\},F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F))$$

is a group isomorphism. By the commutativity of the index maps (Axiom 1.2.8), $\delta_i = \delta_i' \circ K_i(\varphi''')$. Thus if we identify the above groups using δ_i' then δ_i is identified with $K_i(\varphi''')$. By Corollary 2.3.2

$$K_i(\varphi''') : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_i(\mathscr{C}_0(\mathbf{S}_n\setminus\{\omega\},F)), \quad (a,b)\longmapsto a+b\,.$$

Since $\mathbf{S}_n\setminus\{\omega\}$ is homeomorphic to $\mathbb{R}^n$, we get

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F)), \quad (a,b)\longmapsto a+b\,.$$

Thus δ_i is surjective and the other assertions follow from the six-term axiom (Axiom 1.2.7).

c) λ' is K-null since it factorizes through null (Theorem 3.1.2 a)). Put $\omega := (1,0,\cdots,0)\in\mathbb{B}_{n+1}$ and denote by

$$\lambda''' : \mathscr{C}_0(\mathbf{S}_n\setminus\{\omega\},F) \longrightarrow \mathscr{C}(\mathbf{S}_n,F)$$

the inclusion map. By the proof of b), since $\lambda'' = \lambda'''\circ\varphi'''$,

$$K_i\left(\lambda''\right) : K_i\left(\mathscr{C}\left(\Omega',F\right)\right) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_n,F)), \quad (a,b)\longmapsto (0,a+b)$$

by the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

PROPOSITION 3.2.10 *Let Γ be a closed set of $\mathbb{R}^n$, $\Gamma\neq\mathbb{R}^n$,*

$$\varphi : \mathscr{C}_0(\mathbb{R}^n\setminus\Gamma,F) \longrightarrow \mathscr{C}_0(\mathbb{R}^n,F)$$

the inclusion map,

$$\psi : \mathscr{C}_0(\mathbb{R}^n,F) \longrightarrow \mathscr{C}_0(\Gamma,F), \quad x\longmapsto x|\Gamma,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0\longrightarrow \mathscr{C}_0(\mathbb{R}^n\setminus\Gamma,F) \xrightarrow{\varphi} \mathscr{C}_0(\mathbb{R}^n,F) \xrightarrow{\psi} \mathscr{C}_0(\Gamma,F)\longrightarrow 0\,.$$

a) ψ *is K-null.*

b) The sequence

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Gamma,F)) \xrightarrow{\delta_{i+1}} K_i(\mathscr{C}_0(\mathbb{R}^n\setminus\Gamma,F)) \xrightarrow{K_i(\varphi)} \mathscr{C}_0(\Gamma,F) \longrightarrow 0$$

is exact.

c) Let $(\Omega_j)_{j\in J}$ *be a finite family of pairwise disjoint open sets of* $\mathbb{R}^n$ *the union of which is* $\mathbb{R}^n\setminus\Gamma$*. If there is a* $j_0\in J$ *such that* $\mathscr{C}_0\left(\mathbb{R}^n\setminus\Omega_{j_0},F\right)$ *is K-null then for every clopen set* Γ' *of* Γ

$$K_i\left(\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\right)\approx K_{i+1}\left(\mathscr{C}_0\left(\Gamma',F\right)\right)\times K_{i+n}(F)\,.$$

a) follows from Proposition 2.4.10.

b) follows from a) and the six-term axiom (Axiom 1.2.7).

c) We use the notation of Proposition 2.3.7. For $\Gamma'=\Gamma$ the assertion follows from Proposition 2.3.7 c_2) and Theorem 3.2.2 b). Let

$$\tilde{\varphi}:\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\longrightarrow\mathscr{C}_0(\mathbb{R}^n,F)\,,$$

$$\tilde{\tilde{\varphi}}:\mathscr{C}_0(\mathbb{R}^n\setminus\Gamma,F)\longrightarrow\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)$$

be the inclusion maps,

$$\tilde{\psi}:\mathscr{C}_0(\mathbb{R}^n,F)\longrightarrow\mathscr{C}_0\left(\Gamma',F\right)\,,\quad x\longmapsto x|\Gamma'\,,$$

$\tilde{\delta}_i$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\xrightarrow{\tilde{\varphi}}\mathscr{C}_0(\mathbb{R}^n,F)\xrightarrow{\tilde{\psi}}\mathscr{C}_0\left(\Gamma',F\right)\longrightarrow 0\,,$$

and $\tilde{\Phi}_i:=K_i\left(\tilde{\tilde{\varphi}}\right)\circ\Phi_i$. Since $\varphi=\tilde{\varphi}\circ\tilde{\tilde{\varphi}}$,

$$K_i(\tilde{\varphi})\circ\tilde{\Phi}_i=K_i(\tilde{\varphi})\circ K_i\left(\tilde{\tilde{\varphi}}\right)\circ\Phi_i=K_i(\varphi)\circ\Phi_i=id_{K_i(\mathscr{C}_0(\mathbb{R}^n,F))}\,.$$

Thus

$$0\longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\Gamma',F\right)\right)\xrightarrow{\tilde{\delta}_{i+1}}K_i\left(\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\right)\underset{\tilde{\Phi}_i}{\overset{K_i(\tilde{\varphi})}{\rightleftarrows}}K_i(\mathscr{C}_0(\mathbb{R}^n,F))\longrightarrow 0$$

is a split exact sequence and this implies c). ∎

PROPOSITION 3.2.11 *Let Ω, Ω' be compact spaces and $m \in \mathbb{N}$. If Ω is path connected, $\Omega \times \Omega' \subset \mathbb{B}_n$, and $\mathbb{B}_n \setminus (\Omega \times \Omega')$ is homeomorphic to the topological sum of $\mathbb{B}_n \setminus (\Omega \times \mathbb{B}_m)$ and $\Omega \times (\mathbb{B}_m \setminus \Omega')$ then for all $\omega \in \Omega$ and $\omega_0 \in \Omega \times \Omega'$*

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) .$$

In particular if there is a $p \in \mathbb{N}$ such that $\mathbb{B}_m \setminus \Omega'$ is homeomorphic to p copies of $\mathbb{R}^m$ then

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)^p .$$

By Theorem 3.1.2 b) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus (\Omega \times \Omega'), F\right)\right) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus (\Omega \times \mathbb{B}_m), F\right)\right) \times K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) .$$

By Theorem 3.1.2 b) and Corollary 3.1.5,

$$K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus (\Omega \times \mathbb{B}_m), F\right)\right) \approx K_i\left(\mathscr{C}_0\left((\Omega \times \mathbb{B}_m) \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right)$$

and so

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) .$$

We prove now the last assertion. By Theorem 3.1.2 a),

$$K_{i+1}\left(\mathscr{C}_0\left(\mathbb{R}^m, \mathscr{C}\left(\Omega, F\right)\right)\right) \approx K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)$$

so by the Product Theorem (Proposition 2.3.1 a)),

$$K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) \approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_m \setminus \Omega', \mathscr{C}\left(\Omega, F\right)\right)\right) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{R}^m, \mathscr{C}\left(\Omega, F\right)\right)\right)^p \approx K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)^p ,$$

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)^p . \qquad \blacksquare$$

COROLLARY 3.2.12 *Let Ω be a connected graph contained in $\mathbb{B}_2$ and containing $\mathbb{S}_1$, r_0 and r_1 the number of vertices and chords of Ω, respectively, and Γ a nonempty finite subset of $\mathbb{S}_n \times \Omega$. Then*

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \Gamma, F)) \approx$$

$$\approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{r_1-r_0+Card\,\Gamma}\ .$$

Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \mathbb{S}_n \times \Omega$. There is an embedding of $\mathbb{S}_n \times \Omega$ in $\mathbb{B}_{n+2}$ such that $\mathbb{B}_{n+2} \setminus (\mathbb{S}_n \times \Omega)$ is homeomorphic to the topological sum of $\mathbb{B}_{n+2} \setminus (\mathbb{S}_n \times \mathbb{B}_2)$ and $\mathbb{S}_n \times (\mathbb{B}_2 \setminus \Omega)$. Since $\mathbb{S}_n \times (\mathbb{B}_2 \setminus \Omega)$ is homeomorphic to $1 - r_0 + r_1$ copies of $\mathbb{S}_n \times \mathbb{R}^2$, we get by Proposition 3.2.11, for $\omega \in \mathbb{S}_n$,

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \approx$$

$$\approx K_i(\mathscr{C}_0(\mathbb{S}_n \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}(\mathbb{S}_n, F))^{1-r_0+r_1}\ .$$

By Theorem 3.2.2 a),b),

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{1-r_0+r_1}.$$

By Proposition 2.4.11,

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \Gamma, F)) \approx$$

$$\approx K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{Card\,\Gamma-1} \approx$$

$$\approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{r_1-r_0+Card\,\Gamma}\ . \qquad \blacksquare$$

COROLLARY 3.2.13 *If*

$$\Omega := \mathbb{S}_{n-1} \cup \left(\bigcup_{j \in \mathbb{N}_n} \{\, \alpha \in \mathbb{B}_n \mid \alpha_j = 0 \,\} \right),$$

$m \in \mathbb{N}$, and Γ is a finite subset of $\mathbb{S}_m \times \Omega$ then

$$K_i(\mathscr{C}_0((\mathbb{S}_m \times \Omega) \setminus \Gamma, F)) \approx$$

$$\approx K_{i+m}(F) \times K_{i+n+1}(F)^{2^n} \times K_{i+m+n+1}(F)^{2^n} \times K_{i+1}(F)^{Card\,\Gamma-1}\ .$$

Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \mathbf{S}_m \times \Omega$. There is an embedding of $\mathbf{S}_m \times \Omega$ in $\mathbb{B}_{m+n+1}$ such that $\mathbb{B}_{m+n+1} \setminus (\mathbf{S}_m \times \Omega)$ is homeomorphic to the topological sum of $\mathbb{B}_{m+n+1} \setminus (\mathbf{S}_m \times \mathbb{B}_n)$ and $\mathbf{S}_m \times (\mathbb{B}_n \setminus \Omega)$. Since $\mathbb{B}_n \setminus \Omega$ is homeomorphic to the topological sum of 2^n copies of $\mathbb{R}^n$, by Proposition 3.2.11, for $\omega \in \mathbf{S}_m$,

$$K_i\left(\mathscr{C}_0\left((\mathbf{S}_m \times \Omega) \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\mathbf{S}_m \setminus \{\omega\}, F\right)\right) \times K_{i+n+1}\left(\mathscr{C}\left(\mathbf{S}_m, F\right)\right)^{2^n} .$$

By Proposition 3.2.2 a),b),

$$K_i\left(\mathscr{C}_0\left((\mathbf{S}_n \times \Omega) \setminus \{\omega_0\}, F\right)\right) \approx K_{i+m}(F) \times K_{i+1+n}(F)^{2^n} \times K_{i+1+m+n}(F)^{2^n} .$$

By Proposition 2.4.11,

$$K_i\left(\mathscr{C}_0\left((\mathbf{S}_m \times \Omega) \setminus \Gamma, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left((\mathbf{S}_m \times \Omega) \setminus \{\omega_0\}, F\right)\right) \times K_{i+1}(F)^{Card\,\Gamma - 1} \approx$$

$$\approx K_{i+m}(F) \times K_{i+n+1}(F)^{2^n} \times K_{i+m+n+1}(F)^{2^n} \times K_{i+1}(F)^{Card\,\Gamma - 1} . \qquad \blacksquare$$

LEMMA 3.2.14 *Let $(k_j)_{j \in \mathbb{N}_n}$ be a family in $\mathbb{N}$, $n \neq 1$, and $m := 1 + \sum_{j \in \mathbb{N}_n} k_j$. There is an embedding of $\prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$ in $\mathbb{B}_m$ such that $\mathbb{B}_m \setminus \prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$ has two connected components: one is homeomorphic to $\mathbb{R}^{1+k_n} \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}$ and the other is homeomorphic to $\mathbb{B}_m \setminus \left(\mathbb{B}_{1+k_n} \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}\right)$.*

We prove the assertion by induction with respect to $n \in \mathbb{N} \setminus \{1\}$. Assume first $n = 2$, put

$$\Gamma := \left\{ \alpha \in \mathbb{B}_m \mid \|\alpha\| = \frac{1}{2},\ \alpha_{2+k_1} = \alpha_{3+k_1} = \cdots = \alpha_m = 0 \right\},$$

and for every $\alpha \in \mathbb{B}_m$ denote by $d(\alpha)$ the distance of α to Γ. Then

$$\left\{ \alpha \in \mathbb{B}_m \mid d(\alpha) = \frac{1}{4} \right\}$$

is an embedding of $\mathbf{S}_{k_1} \times \mathbf{S}_{k_2}$ in $\mathbb{B}_m$ with the desired properties.

Let now $n>2$ and assume the assertion holds for $n-1$. Let Γ be a closed set of $\mathbb{B}_{m-k_n}$ homeomorphic to $\prod\limits_{j\in\mathbb{N}_{n-1}}\mathbb{S}_{k_j}$. We may assume $\Gamma\subset\mathbb{S}_{m-k_m}$. We denote for every $\alpha\in\mathbb{B}_m$ by $d(\alpha)$ the distance of α to $\frac{1}{2}\Gamma$. Then $\left\{\,\alpha\in\mathbb{B}_m\mid d(\alpha)=\frac{1}{4}\,\right\}$ is an embedding with the desired properties. ■

PROPOSITION 3.2.15 *Let $(k_j)_{j\in\mathbb{N}_n}$ be a family in $\mathbb{N}$.*

a)

$$\prod_{j=1}^{n}\mathbb{S}_{k_j}\in\Upsilon\,,\qquad \mathbb{R}_\Upsilon\subset\left(\prod_{j=1}^{n}\mathbb{S}_{k_j}\right)_\Upsilon\,,$$

$$K_i\left(\mathscr{C}\left(\prod_{j=1}^{n}\mathbb{S}_{k_j},F\right)\right)\approx$$

$$\approx\begin{cases} K_i(F)^{2^n} & \text{if}\quad \text{all } (k_j)_{j\in\mathbb{N}_n} \text{ are even}\\ \Big(K_i(F)\times K_{i+1}(F)\Big)^{2^{n-1}} & \text{if}\quad \text{not all } (k_j)_{j\in\mathbb{N}_n} \text{ are even}\end{cases}.$$

b) If Γ is a nonempty finite subset of $\prod\limits_{j\in\mathbb{N}_n}\mathbb{S}_{k_j}$ then

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathbb{N}_n}\mathbb{S}_{k_j}\setminus\Gamma,F\right)\right)\approx$$

$$\approx\begin{cases} K_i\,(F)^{2^n-1}\times K_{i+1}\,(F)^{Card\,\Gamma-1} & \text{if}\quad \text{all } k_j \text{ are even}\\ K_i\,(F)^{2^{n-1}-1}\times K_{i+1}\,(F)^{2^{n-1}+Card\,\Gamma-2} & \text{if}\quad \text{not all } k_j \text{ are even}\end{cases}.$$

a) By Theorem 3.2.2 b), $\mathbb{S}_{k_j}\in\Upsilon,\mathbb{R}_\Upsilon\subset(\mathbb{S}_{k_j})_\Upsilon$ for every $j\in J$ so by Proposition 1.5.11 a),f), $\prod\limits_{j=1}^{n}\mathbb{S}_{k_j}\in\Upsilon,\mathbb{R}_\Upsilon\subset\left(\prod\limits_{j=1}^{n}\mathbb{S}_{k_j}\right)_\Upsilon$. By Theorem 3.2.2 b), with the notation of Proposition 1.5.11 a),f),

$$p_j=\frac{3+(-1)^{k_j}}{2}\,,\quad q_j=\frac{1-(-1)^{k_j}}{2}\,,\quad p_j+q_j=2\,,\quad p_j-q_j=1+(-1)^{k_j}\,,$$

$$p_J=\frac{1}{2}\left(2^n+\prod_{j=1}^{n}\left(1+(-1)^{k_j}\right)\right)\,,\quad q_J=\frac{1}{2}\left(2^n-\prod_{j=1}^{n}\left(1+(-1)^{k_j}\right)\right)\,,$$

and this implies the result.

b) Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \prod_{j\in\mathbb{N}_n} \mathbb{S}_{k_j}$. We prove the assertion by induction with respect to $n \in \mathbb{N}$. For $n = 1$ this follows from Theorem 3.2.2 e_1). Let $n \neq 1$ and assume the assertion holds for $n-1$. By Lemma 3.2.14, $\mathbb{B}_m \setminus \prod_{j\in\mathbb{N}_n} \mathbb{S}_{k_j}$ is homeomorphic to the topological sum of $\mathbb{R}^{1+k_n} \times \prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j}$ and $\mathbb{B}_m \setminus \left(\mathbb{B}_{1+k_n} \times \prod_{j\in\mathbb{N}_{n-i}} \mathbb{S}_{k_j}\right)$. By Proposition 3.2.11, for $\omega \in \prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j}$,

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathbb{N}_n} \mathbb{S}_{k_j} \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j} \setminus \{\omega\}, F\right)\right) \times$$

$$\times K_{i+1}\left(\mathscr{C}_0\left((\mathbb{B}_m \setminus \mathbb{S}_{k_n}) \times \prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j}, F\right)\right).$$

By a) and Theorem 3.2.2 g),

$$K_{i+1}\left(\mathscr{C}_0\left((\mathbb{B}_n \setminus \mathbb{S}_{k_n}) \times \prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j}, F\right)\right) \approx$$

$$\approx K_{i+k_n}\left(\mathscr{C}\left(\prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j}, F\right)\right) \approx$$

$$\approx \begin{cases} K_{i+k_n}(F)^{2^{n-1}} & \text{if} \quad \text{all } (k_j)_{j\in\mathbb{N}_{n-1}} \text{ are even} \\ \left(K_i(F) \times K_{i+1}(F)\right)^{2^{n-2}} & \text{if} \quad \text{not all } (k_j)_{j\in\mathbb{N}_{n-1}} \text{ are even} \end{cases}.$$

By the induction hypothesis,

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j} \setminus \{\omega\}, F\right)\right) \approx$$

$$\approx \begin{cases} K_i(F)^{2^{n-1}-1} & \text{if} \quad \text{all } (k_j)_{j\in\mathbb{N}_{n-1}} \text{ are even} \\ K_i(F)^{2^{n-2}-1} \times K_{i+1}(F)^{2^{n-2}} & \text{if} \quad \text{not all } (k_j)_{j\in\mathbb{N}_{n-1}} \text{ are even} \end{cases}$$

so

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathbb{N}_n} \mathbb{S}_{k_j} \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx \begin{cases} K_i(F)^{2^n-1} & \text{if} \quad \text{all } (k_j)_{j\in\mathbb{N}_n} \text{ are even} \\ K_i(F)^{2^{n-1}-1} \times K_{i+1}(F)^{2^{n-1}} & \text{if} \quad \text{not all } (k_j)_{j\in\mathbb{N}_n} \text{ are even} \end{cases} .$$

This finishes the inductive proof.

We prove now the general case and put $\Omega := \prod_{j\in\mathbb{N}_n} \mathbf{S}_{k_j}$. Since it is possible to find a closed set Δ of Ω such that $\Gamma \subset \Delta$ and $\Delta \setminus \{\omega_0\}$ is K-null, the assertion follows from Proposition 2.4.11. ■

3.3 Some Morphisms

PROPOSITION 3.3.1 *We put*

$$\vartheta : \mathbb{B}_n \longrightarrow \mathbb{B}_n, \quad (\alpha_j)_{j\in\mathbb{N}_n} \longmapsto (\alpha_1, \cdots, \alpha_{n-1}, -\alpha_n),$$

$$\vartheta' : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad (\alpha_j)_{j\in\mathbb{N}_n} \longmapsto (\alpha_1, \cdots, \alpha_{n-1}, -\alpha_n),$$

$$\vartheta'' : \mathbf{S}_{n-1} \longrightarrow \mathbf{S}_{n-1}, \quad (\alpha_j)_{j\in\mathbb{N}_n} \longmapsto (\alpha_1, \cdots, \alpha_{n-1}, -\alpha_n),$$

$$\phi : \mathscr{C}(\mathbb{B}_n, F) \longrightarrow \mathscr{C}(\mathbb{B}_n, F), \quad x \longmapsto x \circ \vartheta,$$

$$\phi' : \mathscr{C}_0(\mathbb{R}^n, F) \longrightarrow \mathscr{C}_0(\mathbb{R}^n, F), \quad x \longmapsto x \circ \vartheta',$$

$$\phi'' : \mathscr{C}(\mathbf{S}_{n-1}, F) \longrightarrow \mathscr{C}(\mathbf{S}_{n-1}, F), \quad x \longmapsto x \circ \vartheta''.$$

a) $K_i(\phi) : K_i(\mathscr{C}(\mathbb{B}_n, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{B}_n, F)), \quad a \longmapsto a.$

b) $K_i(\phi') : K_i(\mathscr{C}_0(\mathbb{R}^n, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{R}^n, F)), \quad b \longmapsto -b.$

c)

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbf{S}_{n-1}, F)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1}, F)),$$

$$(a,b) \longmapsto \begin{cases} (b,a) & \text{if} \quad n = 1 \\ (a,-b) & \text{if} \quad n > 1 \end{cases},$$

where we identified $K_i(\mathscr{C}(\mathbf{S}_{n-1}, F))$ *with*

$$K_i(\mathscr{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$$

using the group isomorphism of Corollary 3.2.8 d) *if* $n > 1$.

a) follows from the homotopy axiom (Axiom 1.2.5) since ϕ is homotopic to the identity map of $\mathscr{C}(\mathrm{I\!B}_n, F)$.

b) We identify $\mathrm{I\!R}^n$ with the homeomorphic space $\mathrm{I\!B}_n \setminus \mathbf{S}_{n-1}$.

Assume first $n = 1$. Put

$$\psi : \mathscr{C}(\mathrm{I\!B}_1, F) \longrightarrow \mathscr{C}(\{-1,1\}, F)\,, \quad x \longmapsto x|\{-1,1\}$$

and denote by $\varphi : \mathscr{C}_0(]-1,1[, F) \longrightarrow \mathscr{C}(\mathrm{I\!B}_1, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(]-1,1[, F) \xrightarrow{\varphi} \mathscr{C}(\mathrm{I\!B}_1, F) \xrightarrow{\psi} \mathscr{C}(\{-1,1\}, F) \longrightarrow 0\,.$$

By Corollary 2.4.7, $K_i(\psi)a = (a,a)$ for every $a \in K_i(\mathscr{C}(\mathrm{I\!B}_1, F))$ so by the six-term axiom (Axiom 1.2.7),

$$\delta_i(a+b, a+b) = 0\,, \qquad \delta_i(a,b) = -\delta_i(b,a)$$

for all $(a,b) \in K_i(\mathscr{C}(\{-1,1\}, F))$. By the commutativity of the index maps (Axiom 1.2.8), $K_{i+1}(\phi') \circ \delta_i = \delta_i \circ K_i(\phi'')$. For $(a,b) \in K_i(\mathscr{C}(\{-1,1\}, F))$, by the above,

$$K_{i+1}(\phi')\delta_i(a,b) = \delta_i K_i(\phi'')(a,b) = \delta_i(b,a) = -\delta_i(a,b)\,.$$

Since δ_i is surjective (because φ factorizes through null and is therefore K-null), $K_i(\phi')b = -b$ for all $b \in K_i(\mathscr{C}_0(]-1,1[, F))$.

If $n > 1$ then the assertion follows from the case $n = 1$, since $\mathscr{C}_0(\mathrm{I\!R}^n, F) \approx \mathscr{C}_0\left(\mathrm{I\!R}, \mathscr{C}_0\left(\mathrm{I\!R}^{n-1}, F\right)\right)$

c) follows from a), b), and Corollary 3.2.8 c). ■

COROLLARY 3.3.2 *If we put*

$$\vartheta : \mathrm{I\!B}_n \longrightarrow \mathrm{I\!B}_n\,, \quad \alpha \longmapsto -\alpha\,,$$

$$\vartheta' : \mathrm{I\!R}^n \longrightarrow \mathrm{I\!R}^n\,, \quad \alpha \longmapsto -\alpha\,,$$

$$\vartheta'' : \mathbf{S}_{n-1} \longrightarrow \mathbf{S}_{n-1}\,, \quad \alpha \longmapsto -\alpha\,,$$

$$\phi : \mathscr{C}(\mathrm{I\!B}_n, F) \longrightarrow \mathscr{C}(\mathrm{I\!B}_n, F)\,, \quad x \longmapsto x \circ \vartheta\,,$$

$$\phi' : \mathscr{C}_0(\mathbb{R}^n,F) \longrightarrow \mathscr{C}_0(\mathbb{R}^n,F)\,, \quad x \longmapsto x\circ\vartheta'\,,$$

$$\phi'' : \mathscr{C}(\mathbf{S}_{n-1},F) \longrightarrow \mathscr{C}(\mathbf{S}_{n-1},F)\,, \quad x \longmapsto x\circ\vartheta''$$

then

$$K_i(\phi) : K_i(\mathscr{C}(\mathbb{B}_n,F)) \longrightarrow K_i(\mathscr{C}(\mathbb{B}_n,F))\,, \quad a \longmapsto a\,,$$

$$K_i(\phi') : K_i(\mathscr{C}_0(\mathbb{R}^n,F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{R}^n,F))\,, \quad b \longmapsto (-1)^n b\,,$$

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbf{S}_{n-1},F)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1},F))\,,$$

$$(a,b) \longmapsto \begin{cases} (b,a) & \text{if} \quad n=1 \\ (a,(-1)^{n+1}b) & \text{if} \quad n>1 \end{cases}\,,$$

where we identified $K_i(\mathscr{C}(\mathbf{S}_{n-1},F))$ *with*

$$K_i(\mathscr{C}(\mathbb{B}_n,F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1},F))$$

using the group isomorphism of Corollary 3.2.8 c) *if* $n>1$.

The assertion for $K_i(\phi)$ follows from the homotopy axiom (Axiom 1.2.5) since ϕ is homotopic to the identity map of $\mathscr{C}(\mathbb{B}_n,F)$. If n is even then the same holds for $K_i(\phi')$. Assume now n odd and let us denote by $\bar{\phi}'$ the map denoted by ϕ' in Proposition 3.3.1. Then $\phi'\circ\bar{\phi}'$ is homotopic to the identity map of $\mathscr{C}_0(\mathbb{R}^n,F)$ so by Corollary 3.3.1, for every $b\in K_i(\mathscr{C}_0(\mathbb{R}^n,F))$,

$$K_i(\phi')b = -K_i(\phi')K_i(\bar{\phi}')b = -b = (-1)^n b\,.$$

The assertion for $K_i(\phi'')$ follows from the corresponding assertions for $K_i(\phi)$ and $K_i(\phi')$ and from Corollary 3.2.8 d). ■

PROPOSITION 3.3.3 *Let* $\alpha,\beta\in[0,2\pi[$, $\alpha<\beta$, $\Omega := \left\{ e^{i\omega} \;\middle|\; \omega\in]\alpha,\beta[\right\}$, $\Gamma := \mathbb{T}\setminus\Omega$, $\varphi : \mathscr{C}_0(\Omega,F) \longrightarrow \mathscr{C}(\mathbb{T},F)$ *the inclusion map, and*

$$\psi : \mathscr{C}(\mathbb{T},F) \longrightarrow \mathscr{C}(\Gamma,F)\,, \quad x \longmapsto x|\Gamma\,,$$

$$\bar{\psi} : \mathscr{C}(\Gamma,F) \longrightarrow F\,, \quad x \longmapsto x(1)\,,$$

$$\vartheta :]0,2\pi[\longrightarrow]\alpha,\beta[\,, \quad \omega \longmapsto \frac{\beta-\alpha}{2\pi}\omega+\alpha\,.$$

For every $x \in \mathscr{C}_0(\Omega, F)$ put

$$\tilde{x} : \mathbb{T} \longrightarrow F, \quad e^{i\omega} \longmapsto \begin{cases} x\left(e^{i\vartheta(\omega)}\right) & \text{if} \quad \omega \in]0, 2\pi[\\ 0 & \text{if} \quad \omega \in \{0, 2\pi\} \end{cases}$$

and define

$$\phi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F), \quad x \longmapsto \tilde{x}.$$

a) $K_i(\phi)$ and $K_i(\bar{\psi})$ are group isomorphisms and so

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(F), \qquad K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F).$$

b) If we identify $K_i(\mathscr{C}_0(\Omega, F))$ with $K_{i+1}(F)$ and $K_i(\mathscr{C}(\Gamma, F))$ with $K_i(F)$ using the isomorphisms from a) and $K_i(\mathscr{C}(\mathbb{T}, F))$ with $K_i(F) \times K_{i+1}(F)$ using e.g. Alexandroff K-theorem (Theorem 2.2.1 a)) *then*

$$K_i(\varphi) : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T}, F)), \quad b \longmapsto (0, b),$$

$$K_i(\psi) : K_i(\mathscr{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)), \quad (a, b) \longmapsto a.$$

a) ϕ is an E-C*-isomorphism. Put

$$\tilde{\psi} : F \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto 1_{\mathscr{C}(\Gamma, \mathbb{C})} x.$$

Then $\mathscr{C}(\Gamma, F) \xrightarrow{\bar{\psi}} F \xrightarrow{\tilde{\psi}} \mathscr{C}(\Gamma, F)$ is a homotopy in $\mathfrak{M}_E$ so $K_i(\phi)$ and $K_i(\bar{\psi})$ are group isomorphisms by the homotopy axiom (Axiom 1.2.5). The last assertion follows now from Theorem 3.2.2 a).

b) For every $s \in [0, 1]$ put

$$\vartheta_s : \mathbb{T} \longrightarrow \mathbb{T}, \quad e^{i\omega} \longmapsto \begin{cases} e^{is\omega} & \text{if} \quad \omega \in [0, \alpha] \\ e^{is\omega} e^{\frac{2\pi i(1-s)(\omega-\alpha)}{\beta-\alpha}} & \text{if} \quad \omega \in]\alpha, \beta[\\ e^{is\omega} e^{2\pi i(1-s)} & \text{if} \quad \omega \in [\beta, 2\pi] \end{cases},$$

$$\phi_s : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F), \quad x \longmapsto x \circ \vartheta_s.$$

Then $(\phi_s)_{s \in [0,1]}$ is a pointwise continuous path in $\mathscr{C}(\mathbb{T}, F)$ such that ϕ_1 is the identity map. By the homotopy axiom (Axiom 1.2.5), $K_i(\phi_0)$ is the identity map of $K_i(\mathscr{C}(\mathbb{T}, F))$. Let

$$\varphi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$$

be the inclusion map and

$$\psi' : \mathscr{C}(\mathbb{T},F) \longrightarrow F\,, \quad x \longmapsto x(1)\,.$$

Then $\phi_0 \circ \varphi = \varphi' \circ \phi$ and $\psi' \circ \phi_0 = \bar{\psi} \circ \psi$ so (by a)) for $a \in K_i(F)$ and $b \in K_{i+1}(F)$,

$$K_i(\varphi)b = K_i(\phi_0)K_i(\varphi)b = K_i(\varphi')K_i(\phi)b = K_i(\varphi')b = (0,b)\,,$$

$$K_i(\psi)(a,b) = K_i(\bar{\psi})K_i(\psi)(a,b) = K_i(\psi')K_i(\phi_0)(a,b) = K_i(\psi')(a,b) = a$$

by the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

PROPOSITION 3.3.4 *Put* $\Gamma := \left\{ e^{\frac{2\pi i j}{n}} \;\middle|\; j \in \mathbb{N}_n \right\}$ *and*

$$\psi : \mathscr{C}(\mathbb{T},F) \longrightarrow \mathscr{C}(\Gamma,F)\,, \quad x \longmapsto x|\Gamma\,,$$

and denote by

$$\varphi : \mathscr{C}_0(\mathbb{T}\setminus\Gamma,F) \longrightarrow \mathscr{C}(\mathbb{T},F)$$

the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{T}\setminus\Gamma,F) \xrightarrow{\varphi} \mathscr{C}(\mathbb{T},F) \xrightarrow{\psi} \mathscr{C}(\Gamma,F) \longrightarrow 0\,.$$

a) $K_i(\mathscr{C}_0(\mathbb{T}\setminus\Gamma,F)) \approx K_{i+1}(F)^n, \quad K_i(\mathscr{C}(\Gamma,F)) \approx K_i(F)^n.$

b) We identify the isomorphic groups of a) and identify $K_i(\mathscr{C}(\mathbb{T},F))$ with $K_i(F) \times K_{i+1}(F)$ (Theorem 3.2.2 b)).

$$K_i(\varphi) : K_i(\mathscr{C}_0(\mathbb{T}\setminus\Gamma,F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T},F))\,, \quad (b_j)_{j\in\mathbb{N}_n} \longmapsto \left(0, \sum_{j\in\mathbb{N}_n} b_j\right),$$

$$K_i(\psi) : K_i(\mathscr{C}(\mathbb{T},F)) \longrightarrow K_i(\mathscr{C}(\Gamma,F))\,, \quad (a,b) \longmapsto (a)_{j\in\mathbb{N}_n}\,.$$

If $n = 2$ and $K_i(F)$ is isomorphic to $\mathbf{Z}$ or to $\mathbf{Z}_p$ for some $p \in \mathbb{N}$ or to the group of rational numbers then there is an automorphism

$$\Phi_i : K_i(F) \longrightarrow K_i(F)$$

such that

$$\delta_i : K_i(\mathscr{C}(\Gamma,F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{T}\setminus\Gamma,F))\,, \quad (a,b) \longmapsto (\Phi_i(a-b), \Phi(b-a)).$$

c) If we put

$$\vartheta : \mathbb{T}\setminus\Gamma \longrightarrow \mathbb{T}\setminus\{1\}, \quad z \longmapsto z^n,$$

$$\vartheta' : \mathbb{T} \longrightarrow \mathbb{T}, \quad z \longmapsto z^n,$$

$$\vartheta'' : \Gamma \longrightarrow \{1\}, \quad z \longmapsto z^n,$$

$$\phi : \mathscr{C}_0(\mathbb{T}\setminus\{1\},F) \longrightarrow \mathscr{C}_0(\mathbb{T}\setminus\Gamma,F), \quad x \longmapsto x\circ\vartheta,$$

$$\phi' : \mathscr{C}(\mathbb{T},F) \longrightarrow \mathscr{C}(\mathbb{T},F), \quad x \longmapsto x\circ\vartheta',$$

$$\phi'' : \mathscr{C}(\{1\},F) \longrightarrow \mathscr{C}(\Gamma,F), \quad x \longmapsto x\circ\vartheta''$$

then, with the identifications of a) and b),

$$K_i(\phi) : K_i(\mathscr{C}_0(\mathbb{T}\setminus\{1\},F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{T}\setminus\Gamma,F)), \quad b \longmapsto (b)_{j\in\mathbb{N}_n},$$

$$K_i(\phi') : K_i(\mathscr{C}(\mathbb{T},F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T},F)), \quad (a,b) \longmapsto (a,nb),$$

$$K_i(\phi'') : K_i(\mathscr{C}(\{1\},F)) \longrightarrow K_i(\mathscr{C}(\Gamma,F)), \quad a \longmapsto (a)_{j\in\mathbb{N}_n}.$$

a) Put $\Omega_j := \left\{ e^{\frac{2\pi i\omega}{n}} \;\middle|\; \omega \in]j-1, j[\right\}$ for every $j \in \mathbb{N}_n$. By Proposition 3.3.3 a), for every $j \in \mathbb{N}_n$,

$$K_i(\mathscr{C}_0(\Omega_j,F)) \approx K_{i+1}(F).$$

so

$$K_i(\mathscr{C}_0(\mathbb{T}\setminus\Gamma,F)) \approx K_{i+1}(F)^n, \qquad K_i(\mathscr{C}(\Gamma,F)) \approx K_i(F)^n$$

by the Product Theorem (Proposition 2.3.1 a)).

b) By Corollary 2.4.7,

$$K_i(\psi) : K_i(\mathscr{C}(\mathbb{T},F)) \longrightarrow K_i(\mathscr{C}(\Gamma,F)), \quad (a,b) \longmapsto (a)_{j\in\mathbb{N}_n}.$$

If we denote by

$$\varphi_j : \mathscr{C}_0(\Omega_j,F) \longrightarrow \mathscr{C}(\mathbb{T},F)$$

the inclusion map then

$$K_i(\varphi_j) : K_i(\mathscr{C}_0(\Omega_j,F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T},F)), \quad b \longmapsto (0,b)$$

by Proposition 3.3.3 b). By Proposition 3.3.3 a) and Corollary 2.3.2,

$$K_i(\varphi) : K_i(\mathscr{C}_0(\mathbb{T}\setminus\Gamma,F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T},F)), \quad (b_j)_{j\in\mathbb{N}_n} \longmapsto \left(0, \sum_{j\in\mathbb{N}_n} b_j\right).$$

In order to prove the last assertion we define $a', b', a'', b'' \in K_i(F)$ by

$$(a', b') := \delta_i(1, 0), \qquad (a'', b'') := \delta_i(0, 1).$$

From

$$0 = \delta_i(1, 1) = (a', b') + (a'', b'') = (a' + a'', b' + b'')$$

we get $a'' = -a'$ and $b'' = -b'$. There are $j, k \in \mathbf{Z}$ such that $\delta_i(j, k) = (1, -1)$. Then

$$(1, -1) = \delta_i(j, k) = (ja', jb') - (ka', kb') = ((j-k)a', (j-k)b'),$$

$$(j-k)a' = 1, \qquad (j-k)b' = -1.$$

Thus a' is invertible in the ring $K_i(F)$ and $a'^{-1} = j - k$. It follows $b' = -a'$. If we put

$$\Phi_i : K_i(F) \longrightarrow K_i(F), \quad c \longmapsto a'c$$

then Φ_i is an automorphism and for all $a, b \in K_i(F)$,

$$\delta_i(a, b) = (a'a, -a'a) - (a'b, -a'b) = (a'(a-b), a'(b-a)) = (\Phi_i(a-b), \Phi_i(b-a)).$$

c) The assertions for $K_i(\phi)$ and $K_i(\phi'')$ follow from the Product Theorem (Proposition 2.3.1 a)). If $\varphi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$ denotes the inclusion map and

$$\psi' : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}(\{1\}, F), \quad x \longmapsto x|\{1\}$$

then the diagram

$$\begin{array}{ccccc}
K_i(\mathscr{C}_0(\mathbb{T} \setminus \{1\}, F)) & \xrightarrow{K_i(\varphi')} & K_i(\mathscr{C}(\mathbb{T}, F)) & \xrightarrow{K_i(\psi')} & K_i(\mathscr{C}(\{1\}, F)) \\
{\scriptstyle K_i(\phi)}\downarrow & & \downarrow{\scriptstyle K_i(\phi')} & & \downarrow{\scriptstyle K_i(\phi'')} \\
K_i(\mathscr{C}_0(\mathbb{T} \setminus \Gamma, F)) & \xrightarrow[K_i(\varphi)]{} & K_i(\mathscr{C}(\mathbb{T}, F)) & \xrightarrow[K_i(\psi)]{} & K_i(\mathscr{C}(\Gamma, F))
\end{array}$$

is commutative. Let $(a, b) \in K_i(\mathscr{C}(\mathbb{T}, F))$ and put $(a', b') := K_i(\phi')(a, 0)$. By b),

$$(a)_{j \in \mathbb{N}_n} = K_i(\phi'')a = K_i(\phi'')K_i(\psi')(a, 0) =$$

$$= K_i(\psi)K_i(\phi')(a, 0) = K_i(\psi)(a', b') = (a')_{j \in \mathbb{N}_n},$$

$$K_i(\phi')(0, b) = K_i(\phi')K_i(\varphi')b = K_i(\varphi)K_i(\phi)b = K_i(\varphi)(b)_{j \in \mathbb{N}_n} = (0, nb)$$

so $K_i(\phi')(a, b) = (a, nb)$. ■

COROLLARY 3.3.5 *If we put*

$$\vartheta : \mathbb{B}_2 \longrightarrow \mathbb{B}_2\,, \quad z \longmapsto z^n\,,$$

$$\vartheta' : \mathbb{C} \longrightarrow \mathbb{C}\,, \quad z \longmapsto z^n\,,$$

$$\vartheta'' : \mathbb{S}_1 \longrightarrow \mathbb{S}_1\,, \quad z \longmapsto z^n\,,$$

$$\phi : \mathscr{C}(\mathbb{B}_2, F) \longrightarrow \mathscr{C}(\mathbb{B}_2, F)\,, \quad x \longmapsto x \circ \vartheta\,,$$

$$\phi' : \mathscr{C}_0(\mathbb{C}, F) \longrightarrow \mathscr{C}_0(\mathbb{C}, F)\,, \quad x \longmapsto x \circ \vartheta'\,,$$

$$\phi'' : \mathscr{C}(\mathbb{S}_1, F) \longrightarrow \mathscr{C}_0(\mathbb{S}_1, F)\,, \quad x \longmapsto x \circ \vartheta''\,.$$

then $K_i(\phi)$ is the identity map of $K_i(\mathscr{C}(\mathbb{B}_2, F))$ and

$$K_i(\phi') : K_i(\mathscr{C}_0(\mathbb{C}, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{C}, F))\,, \quad a \longmapsto na\,,$$

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbb{S}_1, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{S}_1, F))\,, \quad (a,b) \longmapsto (a, nb)\,,$$

We identify the homeomorphic spaces $\mathbb{C}$ and $\mathbb{B}_2 \setminus \mathbb{S}_1$. By Corollary 3.2.8 c),

$$K_i(\mathscr{C}(\mathbb{S}_1, F)) \approx K_i(\mathscr{C}(\mathbb{B}_2, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_2 \setminus \mathbb{S}_1, F))$$

and by Proposition 3.3.4 e),

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbb{S}_1, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{S}_1, F))\,, \quad (a,b) \longmapsto (a, nb)\,.$$

By Corollary 2.2.2 b) and Theorem 3.1.2 a), $K_i(\phi)$ is the identity map of $K_i(\mathscr{C}(\mathbb{B}_2, F))$ and

$$K_i(\phi') : K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \mathbb{S}_1, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \mathbb{S}_1, F))\,, \quad a \longmapsto na\,. \qquad \blacksquare$$

PROPOSITION 3.3.6 *Let $m, n \in \mathbb{N}$ and*

$$\vartheta_1 : \mathbb{T} \longrightarrow \mathbb{T}\,, \quad w \longmapsto w^m\,,$$

$$\vartheta_2 : \mathbb{T} \longrightarrow \mathbb{T}\,, \quad z \longmapsto z^n\,,$$

$$\psi : \mathscr{C}(\mathbb{T} \times \mathbb{T}, F) \longrightarrow \mathscr{C}(\mathbb{T} \times \mathbb{T}, F)\,, \quad x \longmapsto x \circ (\vartheta_1 \times \vartheta_2)\,.$$

We identify $K_i(\mathscr{C}(\mathbb{T}, F'))$ with $K_i(F') \times K_{i+1}(F')$ for all E-C-algebras F' by using the group isomorphism of* Theorem 3.2.2 b). *Let*

$$a \in K_i(\mathscr{C}(\mathbb{T} \times \mathbb{T}, F)) \approx K_i(\mathscr{C}(\mathbb{T}, \mathscr{C}(\mathbb{T}, F))) \approx K_i(\mathscr{C}(\mathbb{T}, F)) \times K_{i+1}(\mathscr{C}(\mathbb{T}, F))$$

and put $a_0 \in K_i(\mathscr{C}(\mathbb{T},F))$, $a_1 \in K_{i+1}(\mathscr{C}(\mathbb{T},F))$ *such that* $a=(a_0,a_1)$ *and* $a_{0,0},\, a_{1,1} \in K_i(F)$ *and* $a_{0,1},\, a_{1,0} \in K_{i+1}(F)$ *such that* $a_0=(a_{0,0},\, a_{0,1})$ *and* $a_1=(a_{1,0},\, a_{1,1})$. *Then*

$$K_i(\psi)=((a_{0,0},\, mna_{1,1}),\, (na_{0,1},\, ma_{1,0}))\,.$$

We put

$$\phi:\mathscr{C}(\mathbb{T},F)\longrightarrow\mathscr{C}(\mathbb{T},F)\,,\quad x\longmapsto x\circ\vartheta_2\,,$$

$$\phi_1:\mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F))\longrightarrow\mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F))\,,\quad x\longmapsto x\circ\vartheta_1\,,$$

$$\phi_2:\mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F))\longrightarrow\mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F))\,,\quad x\longmapsto \phi\circ x\,,$$

By Corollary 3.3.5,

$$K_i(\phi_1)a=(a_0,\, ma_1),\quad K_i(\phi)a_0=(a_{0,0},\, na_{0,1}),\quad K_{i+1}(\phi)a_1=(a_{1,0},\, na_{1,1})$$

so by Corollary 3.2.8 c), d),

$$K_i(\phi_2)K_i(\phi_1)a=(K_i(\phi)a_0,\, K_{i+1}(\phi)ma_1)=((a_{0,0},\, na_{0,1}),\, (ma_{1,0},\, mna_{1,1}))\,.$$

Since $\psi=\phi_2\circ\phi_1$,

$$K_i(\psi)=((a_{0,0},\, mna_{1,1}),\, (na_{0,1},\, ma_{1,0}))\,.$$

■

3.4 Some Non-orientable Compact Spaces

DEFINITION 3.4.1 *We denote by* $\mathbb{P}_n$ *the n-dimensional* **projective space***, which is obtained from* $\mathbb{B}_n$ *by identifying* α *with* $-\alpha$ *for all* $\alpha\in\mathbb{B}_n$ *with* $\|\alpha\|=1$.

PROPOSITION 3.4.2 *Put*

$$\Omega:=\mathbb{P}_{n+1}\setminus\{\,\alpha\in\mathbb{P}_{n+1}\mid\|\alpha\|=1,\,\alpha_{n+1}=0\,\}\,,$$

$$\Omega':=\{\,\alpha\in\Omega\mid\|\alpha\|=1\,\}\,,$$

$$\psi:\mathscr{C}_0(\Omega,F)\longrightarrow\mathscr{C}_0\left(\Omega',F\right)\,,\quad x\longmapsto x|\Omega'$$

and denote by

$$\varphi:\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbb{S}_n,F)\longrightarrow\mathscr{C}_0(\Omega,F)$$

the inclusion map and by δ_i *the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbb{S}_n,F)\xrightarrow{\varphi}\mathscr{C}_0(\Omega,F)\xrightarrow{\psi}\mathscr{C}_0\left(\Omega',F\right)\longrightarrow 0\,.$$

a) $K_i(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus \mathbf{S}_n,F)) \approx K_{i+n+1}(F)$, $K_i(\mathscr{C}_0(\Omega',F)) \approx K_{i+n}(F)$, *and there is an automorphism* $\Phi_i : K_{i+n}(F) \longrightarrow K_{i+n}(F)$ *such that*

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus \mathbf{S}_n,F)),$$

$$a \longmapsto \Phi_i(a-(-1)^n a) .$$

b) If n is even then $\delta_i = 0$, $K_i(\varphi)$ *is injective,* $K_i(\psi)$ *is surjective, and*

$$\frac{K_i(\mathscr{C}_0(\Omega,F))}{K_{i+1}(F)} \approx K_i(F) .$$

c) If n is odd and for a fixed $i \in \{0,1\}$

$$a \in K_i(F),\, 2a = 0 \Longrightarrow a = 0$$

then $K_i(\psi) = 0$, $K_i(\mathscr{C}_0(\Omega,F)) \approx \frac{K_i(F)}{2K_i(F)}$,

$$K_i(\varphi) : K_i(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus \mathbf{S}_n,F)) \longrightarrow K_i(\mathscr{C}_0(\Omega,F))$$

is the quotient map, and

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus \mathbf{S}_n,F)), \quad a \longmapsto 2\Phi_i a .$$

a) By Theorem 3.2.2 a), $K_i(\mathscr{C}_0(\mathbb{R}^n,F)) \approx K_{i+n}(F)$. Since $\mathbb{B}_{n+1}\setminus \mathbf{S}_n$ is homeomorphic to $\mathbb{R}^{n+1}$, $K_i(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus \mathbf{S}_n,F)) \approx K_{i+n+1}(F)$. Since Ω' is homeomorphic to $\mathbb{R}^n$, $K_i(\mathscr{C}_0(\Omega',F)) \approx K_{i+n}(F)$. We use the notation of Proposition 3.2.9, which we mark by a bar in order to distinguish it from the present notation. Moreover we denote by $\vartheta : \bar{\Omega} \longrightarrow \Omega$ and $\vartheta' : \bar{\Omega}' \longrightarrow \Omega'$ the covering maps and put

$$\phi : \mathscr{C}_0(\Omega,F) \longrightarrow \mathscr{C}_0\left(\bar{\Omega},F\right), \quad x \longmapsto x\circ\vartheta,$$

$$\phi' : \mathscr{C}_0\left(\Omega',F\right) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}',F\right), \quad x \longmapsto x\circ\vartheta' .$$

By the Product Theorem (Proposition 2.3.1 a)), Proposition 3.2.9 a), and Proposition 3.3.1 b),

$$K_i(\phi') : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_i\left(\mathscr{C}_0\left(\bar{\Omega}',F\right)\right), \quad a \longmapsto (a,(-1)^n a) .$$

By the commutativity of the index maps (Axiom 1.2.8), $\delta_i = \bar{\delta}_i \circ K_i(\phi')$ so by Proposition 3.2.9 b),

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus \mathbf{S}_n,F)), \quad a \longmapsto \Phi_i(a-(-1)^n a) .$$

b) and c) follow from a) and the six-term axiom (Axiom 1.2.7). ■

COROLLARY 3.4.3 *We use the notation and the hypothesis of* Proposition 3.4.2, *take* $n=1$, *put* $\Gamma := \{ x \in \mathbb{P}_2 \mid \|x\| = 1 \}$,

$$\psi' : \mathscr{C}(\mathbb{P}_2, F) \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto x|\Gamma,$$

and denote by $\varphi' : \mathscr{C}_0(\mathbb{B}_2 \setminus \mathbf{S}_1, F) \longrightarrow \mathscr{C}(\mathbb{P}_2, F)$ *the inclusion map and by* δ_i' *the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{B}_2 \setminus \mathbf{S}_1, F) \xrightarrow{\varphi'} \mathscr{C}(\mathbb{P}_2, F) \xrightarrow{\psi'} \mathscr{C}(\Gamma, F) \longrightarrow 0 .$$

Then $K_i(\mathscr{C}(\mathbb{P}_2, F)) \approx K_i(F) \times \frac{K_i(F)}{2K_i(F)}$, $K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F) \times K_{i+1}(F)$,

$$K_i(\varphi') : K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \mathbf{S}_1, F)) \longrightarrow K_i(K_i(\mathscr{C}(\mathbb{P}, F))), \quad a \longmapsto (0, \Phi_i a),$$

$$K_i(\psi') : K_i(\mathscr{C}(\mathbb{P}_2, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)), \quad (a, c) \longmapsto (a, 0),$$

$$\delta_i' : K_i(\mathscr{C}(\Gamma, F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_2 \setminus \mathbf{S}_1, F)), \quad (a, b) \longmapsto 2b . \qquad \blacksquare$$

PROPOSITION 3.4.4 *Let*

$$\vartheta : [0,1] \longrightarrow \mathbb{T}, \quad \omega \longmapsto e^{2\pi i \omega},$$

$$\phi : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}([0,1], F), \quad x \longmapsto x \circ \vartheta .$$

If we identify $K_i(\mathscr{C}(\mathbb{T}, F))$ *with* $K_i(F) \times K_{i+1}(F)$ (Theorem 3.2.2 b)) *and* $K_i(\mathscr{C}([0,1], F))$ *with* $K_i(F)$ (Theorem 3.1.2 a)) *then*

$$K_i(\phi) : K_i(\mathscr{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathscr{C}([0,1], F)), \quad (a, b) \longmapsto a .$$

Put

$$\vartheta' :]0,1[\longrightarrow \mathbb{T} \setminus \{1\}, \quad \omega \longmapsto e^{2\pi i \omega},$$

$$\phi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}_0(]0,1[, F), \quad x \longmapsto x \circ \vartheta'$$

and denote by

$$\varphi : \mathscr{C}_0(]0,1[, F) \longrightarrow \mathscr{C}([0,1], F), \quad \varphi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$$

the inclusion maps. Then $\phi \circ \varphi' = \varphi \circ \phi'$, so $K_i(\phi) \circ K_i(\varphi') = K_i(\varphi) \circ K_i(\phi') = 0$, since φ factorizes through 0. Thus $K_i(\phi)(0, b) = 0$ for all $b \in K_{i+1}(F)$.

Put

$$\psi : \mathscr{C}([0,1],F) \longrightarrow \mathscr{C}(\{0,1\},F) \approx F\times F\,, \quad x\longmapsto x|\{0,1\}\,,$$

$$\psi' : \mathscr{C}(\mathbb{T},F) \longrightarrow F\,, \quad x\longmapsto x(1)\,,$$

$$\mu : F \longrightarrow \mathscr{C}(\{0,1\},F)\,, \quad x\longmapsto (x,x)\,.$$

Then $\psi\circ\phi=\mu\circ\psi'$, so $K_i(\psi)\circ K_i(\phi)=K_i(\mu)\circ K_i(\psi')$ and we get (by the above)

$$K_i(\psi)K_i(\phi)(a,b)=K_i(\psi)K_i(\phi)(a,0)=K_i(\mu)K_i(\psi')(a,0)=K_i(\mu)a=(a,a)\,,$$

$$K_i(\phi)(a,b)=a$$

for all $(a,b)\in K_i(F)\times K_{i+1}(F)$. ■

DEFINITION 3.4.5 *We denote by* $\mathbb{M}$ *the* **Möbius band** *obtained from* $[0,1]\times[-1,1]$ *by identifying the points* $(0,\beta)$ *and* $(1,-\beta)$ *for every* $\beta\in[-1,1]$. *We put for every* $j\in\{-1,0,1\}$

$$\Gamma_j^{\mathbb{M}} := \{\ (\alpha,j)\in\mathbb{M}\ |\ \alpha\in[0,1]\,\}\ .$$

PROPOSITION 3.4.6 *For every* $j\in\{-1,0,1\}$ *put*

$$\psi_j : \mathscr{C}(\mathbb{M},F) \longrightarrow \mathscr{C}\left(\Gamma_j^{\mathbb{M}},F\right)\,, \quad x\longmapsto x|\Gamma_j^{\mathbb{M}}\,.$$

a) $\Gamma_0^{\mathbb{M}}$ *is homeomorphic to* $\mathbb{T}$ *and* $\Gamma_j^{\mathbb{M}}$ *is homeomorphic to* $[0,1]$ *for all* $j\in\{-1,1\}$.

b) $\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)$ *is K-null and*

$$K_i(\psi_0) : K_i(\mathscr{C}(\mathbb{M},F)) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma_0^{\mathbb{M}},F\right)\right) \approx K_i(F)\times K_{i+1}(F)$$

is a group isomorphism.

c) If we identify $K_i(\mathscr{C}(\mathbb{M},F))$ *with* $K_i(F)\times K_{i+1}(F)$ *using the group isomorphism* $K_i(\psi_0)$ *of b) and* $K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right)\right)$ *with* $K_i(F)$ *using a)* (and Theorem 3.1.2 a)) *then*

$$K_i(\psi_1) : K_i(\mathscr{C}(\mathbb{M},F)) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right)\right)\,, \quad (a,b)\longmapsto a\,.$$

d) If we put $\omega:=(0,0)=(1,0)\in\mathbb{M}$, $\Gamma:=\{\ (\alpha,0)\ |\ \alpha\in]0,1[\,\}$, *and*

$$\psi : \mathscr{C}_0(\mathbb{M}\setminus\{\omega\},F) \longrightarrow \mathscr{C}_0(\Gamma,F)\,, \quad x\longmapsto x|\Gamma$$

then

$$K_i(\psi) : K_i(\mathscr{C}_0(\mathbb{M}\setminus\{\omega\},F)) \longrightarrow K_i(\mathscr{C}_0(\Gamma,F)) \approx K_{i+1}(F)$$

is a group isomorphism.

e) If Γ' *is a finite subset of* $\mathbb{M}$ *then*

$$K_i\left(\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma',F\right)\right)\approx K_{i+1}(F)^{\Gamma'}$$

a) is easy to see.

b) For every $s\in]0,1]$ put

$$\vartheta_s:\mathbb{M}\setminus\Gamma_0^{\mathbb{M}}\longrightarrow\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},\quad(\alpha,\beta)\longmapsto(\alpha,s\beta)\,.$$

By Proposition 2.4.1 (replacing there Ω by $\mathbb{M}\setminus\Gamma_0^{\mathbb{M}}$), $\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)$ is K-null and the assertion follows from the Topological six-term sequence (Proposition 2.1.8 a)) and a) (and Theorem 3.2.2 b)).

c) follows from b) and Proposition 3.4.4.

d) If $\varphi:\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\{\omega\},F\right)$ denotes the inclusion map then

$$0\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)\xrightarrow{\varphi}\mathscr{C}_0\left(\mathbb{M}\setminus\{\omega\},F\right)\xrightarrow{\psi}\mathscr{C}_0\left(\Gamma,F\right)\longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. By b), $\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 a)), $K_i(\psi)$ is a group isomorphism. Since Γ is homeomorphic to $\mathbb{R}$, $K_i(\mathscr{C}_0\left(\Gamma,F\right))\approx K_{i+1}(F)$ by Theorem 3.2.2 a).

e) follows from d) and Proposition 2.4.11. ■

PROPOSITION 3.4.7 *Put*

$$\Gamma':=\Gamma_0^{\mathbb{M}}\cup\Gamma_1^{\mathbb{M}},\qquad\Gamma'':=\Gamma_0^{\mathbb{M}}\cup\Gamma_1^{\mathbb{M}}\cup\Gamma_{-1}^{\mathbb{M}},\qquad\Gamma''':=\Gamma_1^{\mathbb{M}}\cup\Gamma_{-1}^{\mathbb{M}},$$

$$\mathbb{M}':=\mathbb{M}\setminus\Gamma',\qquad\mathbb{M}'':=\mathbb{M}\setminus\Gamma''\qquad\mathbb{M}''':=\mathbb{M}\setminus\Gamma'''\,.$$

Let

$$\varphi':\mathscr{C}_0\left(\mathbb{M}',F\right)\longrightarrow\mathscr{C}\left(\mathbb{M},F\right),$$

$$\varphi'':\mathscr{C}_0\left(\mathbb{M}'',F\right)\longrightarrow\mathscr{C}\left(\mathbb{M},F\right),$$

$$\bar{\varphi}':\mathscr{C}_0\left(\mathbb{M}',F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right),$$

$$\bar{\varphi}'':\mathscr{C}_0\left(\mathbb{M}'',F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right),$$

$$\varphi''' : \mathscr{C}_0\left(\mathbb{M}'', F\right) \longrightarrow \mathscr{C}_0\left(\mathbb{M}''', F\right),$$

$$\lambda' : \mathscr{C}\left(\Gamma_1^{\mathbb{M}}, F\right) \longrightarrow \mathscr{C}\left(\Gamma', F\right),$$

$$\lambda'' : \mathscr{C}\left(\Gamma''', F\right) \longrightarrow \mathscr{C}\left(\Gamma'', F\right)$$

$$\lambda''' : \mathscr{C}\left(\Gamma_0^{\mathbb{M}}, F\right) \longrightarrow \mathscr{C}\left(\Gamma'', F\right),$$

be the inclusion maps,

$$\psi' : \mathscr{C}\left(\mathbb{M}, F\right) \longrightarrow \mathscr{C}\left(\Gamma', F\right), \quad x \longmapsto x|\Gamma',$$

$$\psi'' : \mathscr{C}\left(\mathbb{M}, F\right) \longrightarrow \mathscr{C}\left(\Gamma'', F\right), \quad x \longmapsto x|\Gamma'',$$

$$\bar{\psi}' : \mathscr{C}_0\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F\right) \longrightarrow \mathscr{C}\left(\Gamma_1^{\mathbb{M}}, F\right), \quad x \longmapsto x|\Gamma_1^{\mathbb{M}},$$

$$\bar{\psi}'' : \mathscr{C}_0\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F\right) \longrightarrow \mathscr{C}\left(\Gamma''', F\right), \quad x \longmapsto x|\Gamma''',$$

$$\psi''' : \mathscr{C}_0\left(\mathbb{M}''', F\right) \longrightarrow \mathscr{C}\left(\Gamma_0^{\mathbb{M}}, F\right), \quad x \longmapsto x|\Gamma_0^{\mathbb{M}},$$

and δ_i', δ_i'', $\bar{\delta}_i'$, $\bar{\delta}_i''$, δ_i''' *the index maps associated to the exact sequences in* $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}', F\right) \xrightarrow{\varphi'} \mathscr{C}\left(\mathbb{M}, F\right) \xrightarrow{\psi'} \mathscr{C}\left(\Gamma', F\right) \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}'', F\right) \xrightarrow{\varphi''} \mathscr{C}\left(\mathbb{M}, F\right) \xrightarrow{\psi''} \mathscr{C}\left(\Gamma'', F\right) \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}', F\right) \xrightarrow{\bar{\varphi}'} \mathscr{C}\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F\right) \xrightarrow{\bar{\psi}'} \mathscr{C}\left(\Gamma_1^{\mathbb{M}}, F\right) \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}'', F\right) \xrightarrow{\bar{\varphi}''} \mathscr{C}\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F\right) \xrightarrow{\bar{\psi}''} \mathscr{C}\left(\Gamma''', F\right) \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}'', F\right) \xrightarrow{\varphi'''} \mathscr{C}_0\left(\mathbb{M}''', F\right) \xrightarrow{\psi'''} \mathscr{C}\left(\Gamma_0^{\mathbb{M}}, F\right) \longrightarrow 0,$$

respectively.

a) Γ''' *is homeomorphic to* $\mathbb{T}$.

b) *The maps*

$$\bar{\delta}_i' : K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{M}}, F\right)\right) \approx K_i(F) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}', F\right)\right),$$

$$\bar{\delta}_i'' : K_i\left(\mathscr{C}\left(\Gamma''', F\right)\right) \approx K_i(F) \times K_{i+1}(F) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}'', F\right)\right)$$

are group isomorphisms.

c) If we put $\Phi_i' := K_i(\lambda') \circ (\bar{\delta}_i')^{-1}$, $\Phi_i'' := K_i(\lambda'') \circ (\bar{\delta}_i'')^{-1}$ *(using b)) then the sequences*

$$0 \longrightarrow K_i(\mathscr{C}(\mathbb{M},F)) \xrightarrow{K_i(\psi')} K_i\left(\mathscr{C}\left(\Gamma',F\right)\right) \underset{\Phi_i'}{\overset{\delta_i'}{\rightleftarrows}} K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}',F\right)\right) \longrightarrow 0\,,$$

$$0 \longrightarrow K_i(\mathscr{C}(\mathbb{M},F)) \xrightarrow{K_i(\psi'')} K_i\left(\mathscr{C}\left(\Gamma'',F\right)\right) \underset{\Phi_i''}{\overset{\delta_i''}{\rightleftarrows}} K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}'',F\right)\right) \longrightarrow 0$$

are split exact and the maps

$$K_i(\mathscr{C}(\mathbb{M},F)) \times K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}',F\right)\right) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma',F\right)\right),$$
$$(a,b) \longmapsto K_i(\psi')a + \Phi_i' b\,,$$
$$K_i(\mathscr{C}(\mathbb{M},F)) \times K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}'',F\right)\right) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma'',F\right)\right),$$
$$(a,b) \longmapsto K_i(\psi'')a + \Phi_i'' b$$

are group isomorphisms.

d) $\delta_i''' = 0$ *and the sequence*

$$0 \longrightarrow K_i\left(\mathscr{C}_0\left(\mathbb{M}'',F\right)\right) \xrightarrow{K_i(\varphi''')} K_i\left(\mathscr{C}_0\left(\mathbb{M}''',F\right)\right)$$

$$K_i\left(\mathscr{C}_0\left(\mathbb{M}''',F\right)\right) \xrightarrow{K_i(\psi''')} K_i\left(\mathscr{C}\left(\Gamma_0^{\mathbb{M}},F\right)\right) \longrightarrow 0$$

is exact.

a) is easy to see.

b) By Proposition 3.4.6 b), $\mathscr{C}_0\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F\right)$ is K-null and the assertion follows from a), the Topological six-term sequence (Proposition 2.1.8 b)), and Proposition 3.4.6 a) (and Theorem 3.1.2 a), Theorem 3.2.2 b)).

c) If we put $\Omega_1 := \mathbb{M}$, $\Omega_2 := \mathbb{M} \setminus \Gamma_0^{\mathbb{M}}$, and $\Omega_3 := \mathbb{M}'$ (respectively $\Omega_3 := \mathbb{M}''$) then the assertion follows from the Topological triple (Proposition 2.1.11 a)).

d) By the commutativity of the index maps (Axiom 1.2.8), $\delta_i''' = \delta_i'' \circ K_i(\lambda''')$. By c), $Im\,(\Phi_i'' \circ \delta_i'') \subset Im\,K_i(\lambda'')$. Since $Im\,K_i(\lambda''') = K_i\left(\mathscr{C}\left(\Gamma_0^{\mathbb{M}},F\right)\right)$ we get

$$\Phi_i'' \circ \delta_i''' = \Phi_i'' \circ \delta_i'' \circ K_i(\lambda''') = 0\,.$$

Thus $\delta_i''' = \delta_i'' \circ \Phi_i'' \circ \delta_i''' = 0$ and the assertion follows from the six-term axiom (Axiom 1.2.7). ■

DEFINITION 3.4.8 *We denote by* $\mathbb{K}$ *the* **Klein bottle** *obtained from the Möbius band* $\mathbb{M}$ *by identifying the points* $(\alpha,-1)$ *and* $(\alpha,1)$ *for all* $\alpha\in[0,1]$ *and put for every* $j\in\{0,1\}$

$$\Gamma_j^{\mathbb{K}} := \{\ (\alpha,j)\in\mathbb{K}\ |\ \alpha\in[0,1]\,\}\ .$$

PROPOSITION 3.4.9 *We put* $\mathbb{K}' := \mathbb{K}\setminus\Gamma_0^{\mathbb{K}}$, $\mathbb{K}'' := \mathbb{K}\setminus(\Gamma_0^{\mathbb{K}}\cup\Gamma_1^{\mathbb{K}})$,

$$\psi : \mathscr{C}_0\left(\mathbb{K}',F\right)\longrightarrow\mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right)\,,\quad x\longmapsto x|\Gamma_1^{\mathbb{K}}$$

and denote by $\varphi:\mathscr{C}_0(\mathbb{K}'',F)\longrightarrow\mathscr{C}_0(\mathbb{K}',F)$ *the inclusion map and by* δ_i *the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0\left(\mathbb{K}'',F\right)\xrightarrow{\varphi}\mathscr{C}_0\left(\mathbb{K}',F\right)\xrightarrow{\psi}\mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right)\longrightarrow 0\,.$$

We use the notation of Proposition 3.4.7 *(so* $\Gamma_0^{\mathbb{K}}=\Gamma_0^{\mathbb{M}}$ *and* $\mathbb{K}''=\mathbb{M}''$*).*

a) $\Gamma_0^{\mathbb{K}}$ *and* $\Gamma_1^{\mathbb{K}}$ *are homeomorphic to* $\mathbb{T}$.

b) The map

$$(\bar{\delta}''_{i+1})^{-1} : K_i\left(\mathscr{C}_0\left(\mathbb{M}'',F\right)\right)\longrightarrow K_{i+1}\left(\mathscr{C}\left(\Gamma''',F\right)\right)\approx K_i(F)\times K_{i+1}(F)$$

is a group isomorphism.

c) If we identify $K_i(\mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right))$ *with* $K_i(F)\times K_{i+1}(F)$ *using a) and* Theorem 3.2.2 b) *and* $K_{i+1}(\mathscr{C}_0(\mathbb{K}'',F))$ *with* $K_i(F)\times K_{i+1}(F)$ *using b) then*

$$\delta_i : K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right)\right)\longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{K}'',F\right)\right)\,,\quad (a,b)\longmapsto(a,2b)\,.$$

d) If δ_i *is injective then* ψ *is K-null and* $K_i(\mathscr{C}_0(\mathbb{K}',F))\approx\frac{K_i(F)}{2K_i(F)}$ *and if we denote by*

$$\Phi_i : K_i(F)\longrightarrow\frac{K_i(F)}{2K_i(F)}$$

the quotient map then

$$K_i(\varphi) : K_i\left(\mathscr{C}_0\left(\mathbb{K}'',F\right)\right)\longrightarrow K_i\left(\mathscr{C}_0\left(\mathbb{K}',F\right)\right)\,,\quad (a,b)\longmapsto\Phi_i b\,.$$

a) is easy to see.

b) follows from Proposition 3.4.7 b).

c) We denote by

$$\vartheta : \mathbb{M} \setminus \Gamma_0^{\mathbb{M}} \longrightarrow \mathbb{K}'$$

the covering map, by

$$\vartheta' : \Gamma''' \longrightarrow \Gamma_1^{\mathbb{K}}$$

the map defined by ϑ, and put

$$\phi : \mathscr{C}_0\left(\mathbb{K}', F\right) \longrightarrow \mathscr{C}_0\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F\right), \quad x \longmapsto x \circ \vartheta\,,$$

$$\phi' : \mathscr{C}_0\left(\Gamma_1^{\mathbb{K}}, F\right) \longrightarrow \mathscr{C}_0\left(\Gamma''', F\right), \quad x \longmapsto x \circ \vartheta'\,.$$

With the identifications of Γ''' and $\Gamma_1^{\mathbb{K}}$ with $\mathbb{T}$ (by a) and Proposition 3.4.7 a)),

$$\vartheta' : \Gamma''' \longrightarrow \Gamma_1^{\mathbb{K}}, \quad z \longmapsto z^2\,.$$

By the commutativity of the index maps (Axiom 1.2.8) the diagrams

$$\begin{array}{ccccc}
K_i(\mathscr{C}_0(\mathbb{M}'', F)) & \xrightarrow{K_i(\varphi)} & K_i(\mathscr{C}_0(\mathbb{K}', F)) & \xrightarrow{K_i(\psi)} & K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}}, F\right)\right) \\
\Big\downarrow{=} & & \Big\downarrow{K_i(\phi)} & & \Big\downarrow{K_i(\phi')} \\
K_i(\mathscr{C}_0(\mathbb{M}'', F)) & \xrightarrow[K_i(\bar{\varphi}'')]{} & K_i(\mathscr{C}_0(\mathbb{M}'''), F)) & \xrightarrow[K_i(\bar{\psi}'')]{} & K_i(\mathscr{C}(\Gamma''', F))
\end{array}$$

$$\begin{array}{ccc}
K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}}, F\right)\right) & \xrightarrow{\delta_i} & K_{i+1}(\mathscr{C}_0(\mathbb{M}'', F)) \\
\Big\downarrow{K_i(\phi')} & & \Big\downarrow{=} \\
K_i(\mathscr{C}(\Gamma''', F)) & \xrightarrow[\bar{\delta}_i'']{} & K_{i+1}(\mathscr{C}_0(\mathbb{M}'', F))
\end{array}$$

are commutative. By Proposition 3.3.4 c),

$$K_i(\phi') : K_i\left(\mathscr{C}\left(\Gamma^{\mathbb{K}}, F\right)\right) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma''', F\right)\right), \quad (a,b) \longmapsto (a, 2b)\,.$$

By b),

$$\delta_i : K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}}, F\right)\right) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{K}'', F\right)\right), \quad (a,b) \longmapsto (a, .2b)\,.$$

d) By the six-term axiom (Axiom 1.2.7), ψ is K-null. The other assertions follow from c) and the six-term axiom (Axiom 1.2.7). ■

3.5 Pasting Locally Compact Spaces

PROPOSITION 3.5.1 *Let Ω_1,Ω_2 be locally compact spaces, Γ_1 and Γ_2 closed sets of Ω_1 and Ω_2, respectively, $\vartheta:\Gamma_1\longrightarrow\Gamma_2$ a homeomorphism, Ω' the topological sum of $\Omega_1\setminus\Gamma_1$ and $\Omega_2\setminus\Gamma_2$, Ω the locally compact space obtained from the topological sum of Ω_1 and Ω_2 by identifying the points ω and $\vartheta(\omega)$ for all $\omega\in\Gamma_1$, Γ the closed set of Ω corresponding to the identified Γ_1 and Γ_2 (so $\Omega\setminus\Gamma=\Omega'$), $\varphi:\mathscr{C}_0(\Omega\setminus\Gamma,F)\longrightarrow\mathscr{C}_0(\Omega,F)$ the inclusion map,*

$$\psi:\mathscr{C}_0(\Omega,F)\longrightarrow\mathscr{C}_0(\Gamma,F)\,,\quad x\longmapsto x|\Gamma\,,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0(\Omega\setminus\Gamma,F)\xrightarrow{\varphi}\mathscr{C}_0(\Omega,F)\xrightarrow{\psi}\mathscr{C}_0(\Gamma,F)\longrightarrow 0\,.$$

Let $J:=\{1,2\}$ and for every $j\in J$ let

$$\varphi_j:\mathscr{C}_0(\Omega_j\setminus\Gamma_j,F)\longrightarrow\mathscr{C}_0(\Omega_j,F)\,,$$

$$\varphi'_j:\mathscr{C}_0(\Omega_j\setminus\Gamma_j,F)\longrightarrow\mathscr{C}_0\left(\Omega',F\right)\,,$$

$$\varphi''_j:\mathscr{C}_0(\Omega_j\setminus\Gamma_j,F)\longrightarrow\mathscr{C}_0(\Omega,F)$$

be the inclusion maps,

$$\psi_j:\mathscr{C}_0(\Omega_j,F)\longrightarrow\mathscr{C}_0(\Gamma_j,F)\,,\quad x\longmapsto x|\Gamma_j\,,$$

$$\psi'_j:\mathscr{C}_0\left(\Omega',F\right)\longrightarrow\mathscr{C}_0(\Omega_j\setminus\Gamma_j,F)\,,\quad x\longmapsto x|(\Omega_j\setminus\Gamma_j)\,,$$

and $\delta_{j,i}$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0(\Omega_j\setminus\Gamma_j,F)\xrightarrow{\varphi_j}\mathscr{C}_0(\Omega_j,F)\xrightarrow{\psi_j}\mathscr{C}_0(\Gamma_j,F)\longrightarrow 0\,.$$

a) $\delta_{j,i}=K_{i+1}(\psi'_j)\circ\delta_i$ *for every $j\in J$ and*

$$\delta_i=K_{i+1}(\varphi'_1)\circ\delta_{1,i}+K_{i+1}(\varphi'_2)\circ\delta_{2,i}\,.$$

b) Assume $\mathscr{C}_0(\Omega_1,F)$ K-null.

b_1) $\delta_{1,i}:K_i(\mathscr{C}_0(\Gamma_1,F))\longrightarrow K_{i11}(\mathscr{C}_0(\Omega_1\setminus\Gamma_1,F))$ *is a group isomorphism.*

b_2) δ_i is injective.

b_3) ψ is K-null.

b_4) $K_i(\varphi_2'') : K_i(\mathscr{C}_0(\Omega_2 \setminus \Gamma_2, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F))$ is a group isomorphism.

b_5) If we put

$$\Phi_i := K_i(\varphi_2') \circ K_i(\varphi_2'')^{-1} : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right)$$

then the map

$$K_{i+1}(\mathscr{C}_0(\Gamma, F)) \times K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right),$$

$$(a, b) \longmapsto \delta_{i+1} a + \Phi_i b$$

is a group isomorphism.

b_6) If also $\mathscr{C}_0(\Omega_2, F)$ is K-null then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F)),$$

$$K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F))^2 .$$

a) follows from Proposition 2.3.7 a), since ψ_j'' of this Proposition is the identity map in the present case.

b_1) follows from the Topological six-term sequence (Proposition 2.1.8 a)).

b_2) Let $a \in K_i(\mathscr{C}_0(\Gamma, F))$ such that $\delta_i a = 0$. By a), $\delta_{1,i} a = K_{i+1}(\psi_1')\delta_i a = 0$ and by b_1), $a = 0$.

b_3) follows from b_2) and the six-term axiom (Axiom 1.2.7).

b_4) and b_5) follow from b_3) and Proposition 2.3.7 c_1), c_2).

b_6) follows from b_1), b_4), and the Product Theorem (Proposition 2.3.1 a)). ■

COROLLARY 3.5.2 *Let Γ be a locally compact space, $(\Omega_j)_{j \in J}$ a nonempty finite family of locally compact spaces such that $\mathscr{C}_0(\Omega_j, F)$ is K-null for every $j \in J$, and for every $j \in J$ let Γ_j be a closed set of Ω_j and $\vartheta_j : \Gamma \longrightarrow \Gamma_j$ a homeomorphism. Let Ω' the topological sum of the family $(\Omega_j \setminus \Gamma_j)_{j \in J}$, and Ω the locally compact space obtained* *from the topological sum of the family* $(\Omega_j)_{j\in J}$ *by identifying for every* $\omega \in \Gamma$ *all the points* $\vartheta_j(\omega)$ $(j \in J)$*. Then*

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F))^{Card\,J-1},$$

$$K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right) \approx K_i(\mathscr{C}_0(\Omega, F)) \times K_{i+1}(\mathscr{C}_0(\Gamma, F)) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F))^{Card\,J}.$$

We prove the Corollary by induction with respect to $Card\,J$. For $Card\,J \in \{1,2\}$ the assertion follows from Proposition 3.5.1 $b_1)$,$b_5)$,$b_6)$. Let $k \in J$, assume the assertion holds for $J' := J \setminus \{\mathrm{k}\}$, and denote by Ω'' the topological sum of the family $(\Omega_j \setminus \Gamma_j)_{j\in J'}$. By Proposition 3.5.1 $b_4)$,$b_5)$ and the induction hypothesis,

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_i\left(\mathscr{C}_0\left(\Omega'', F\right)\right) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F))^{Card\,J-1},$$

$$K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F)) \times K_i\left(\mathscr{C}_0\left(\Omega'', F\right)\right) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F))^J. \qquad \blacksquare$$

COROLLARY 3.5.3 *Let* $m, n \in \mathbb{N}$,

$$\Gamma_+ := \{\, \alpha \in \mathbb{B}_n \mid \|\alpha\| = 1,\, \alpha_n > 0 \,\},\, \Gamma_- := \{\, \alpha \in \mathbb{B}_n \mid \|\alpha\| = 1,\, \alpha_n \le 0 \,\},$$

and Ω *the locally compact space obtained from the topological sum of the family* $(\mathbb{B}_n \setminus \Gamma_-)_{j\in\mathbb{N}_m}$ *by identifying all the* Γ_+*. Then*

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+n}(F)^{m-1}.$$

By Proposition 2.4.1, $\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma_-, F)$ is null-homotopic and so K-null. For $n > 1$, Γ_+ is homeomorphic to $\mathbb{R}^{n-1}$ so by Theorem 3.2.2 a),

$$K_i(\mathscr{C}_0(\Gamma_+, F)) \approx K_{i+n-1}(F)$$

and this relation obviously holds also for $n = 1$. Then by Corollary 3.5.2,

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+n}(F)^{m-1}. \qquad \blacksquare$$

Remark. The above result can be deduced also from Example 2.4.9 by using Proposition 1.5.11 d).

COROLLARY 3.5.4 *Let Ω', Ω'' be locally compact spaces, $\omega' \in \Omega'$, $\omega'' \in \Omega''$, and Ω the locally compact space obtained from the topological sum of Ω' and Ω'' by identifying ω' and ω''. If $\mathscr{C}_0(\Omega'', F)$ is K-null then*

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_i\left(\mathscr{C}_0\left(\Omega' \setminus \{\omega'\}, F\right)\right) .$$

The assertion follows from Proposition 3.5.1 b_4). ∎

PROPOSITION 3.5.5 *Let Ω', Ω'' be compact spaces, $\omega' \in \Omega'$, $\omega'' \in \Omega''$, and Ω the compact space obtained by identifying the points ω' and ω'' in the topological sum of Ω' and Ω''. Then*

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \Omega', F\right)\right) \times K_i\left(\mathscr{C}\left(\Omega', F\right)\right) .$$

Let $\varphi : \mathscr{C}_0(\Omega \setminus \Omega', F) \longrightarrow \mathscr{C}(\Omega, F)$ be the inclusion map and

$$\psi : \mathscr{C}(\Omega, F) \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad x \longmapsto x|\Omega' .$$

We put for every $x \in \mathscr{C}(\Omega', F)$,

$$\lambda x : \Omega \longrightarrow F, \quad \omega \longmapsto \begin{cases} x(\omega) & \text{if} \quad \omega \in \Omega' \\ x(\omega_0) & \text{if} \quad \omega \in \Omega'' \end{cases} ,$$

where $\omega_0 \in \Omega$ denotes the point corresponding to the identified points ω' and ω''. Then

$$0 \longrightarrow \mathscr{C}_0\left(\Omega \setminus \Omega', F\right) \xrightarrow{\varphi} \mathscr{C}(\Omega, F) \underset{\xleftarrow[\lambda]{}}{\xrightarrow{\psi}} \mathscr{C}\left(\Omega', F\right) \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ and the assertion follows from the split exact axiom (Axiom 1.2.3). ∎

PROPOSITION 3.5.6 *Let $(\Omega_j)_{j \in \mathbb{N}_n}$ be a family of compact spaces and for every $j \in \mathbb{N}_n$ let ω_j, ω_j' be distinct points of Ω_j. If Ω denotes the compact space obtained from the topological sum of the family $(\Omega_j)_{j \in \mathbb{N}_n}$ by identifying ω_j' with ω_{j+1} for all $j \in \mathbb{N}_{n-1}$ then*

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times \prod_{j=1}^{n} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_j\}, F\right)) .$$

If $(k_j)_{j \in \mathbb{N}_n}$ is a family in $\mathbb{N}$, $\Omega_j = \mathbf{S}_{k_j}$ for every $j \in \mathbb{N}_n$, and

$$p := Card\left\{ j \in \mathbb{N}_n \mid k_j \text{ is even} \right\}, \qquad q := Card\left\{ j \in \mathbb{N}_n \mid k_j \text{ is odd} \right\}$$

then

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)^{p+1} \times K_{i+1}(F)^q \,.$$

We put $\bar{\Omega}_n := \Omega$ and prove the assertion by induction with respect to $n \in \mathbb{N}$. For $n=1$ the assertion follows from the Alexandroff K-theorem (Theorem 2.2.1 a)). Assume the assertion hods for an $n \in \mathbb{N}$. By Proposition 3.5.5 and the induction hypothesis,

$$K_i\left(\mathscr{C}\left(\bar{\Omega}_{n+1},F\right)\right) \approx K_i\left(\mathscr{C}_0\left(\bar{\Omega}_{n+1}\setminus\bar{\Omega}_n,F\right)\right) \times K_i\left(\mathscr{C}\left(\bar{\Omega}_n,F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_{n+1}\setminus\{\omega_{\mathrm{n+1}}\},F)) \times K_i\left(\mathscr{C}\left(\bar{\Omega}_n,F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_{n+1}\setminus\{\omega_{\mathrm{n+1}}\},F)) \times K_i(F) \times \prod_{j=1}^{n} K_i(\mathscr{C}_0\left(\Omega_j\setminus\{\omega_{\mathrm{j}}\},F\right)) \approx$$

$$\approx K_i(F) \times \prod_{j=1}^{n+1} K_i(\mathscr{C}_0\left(\Omega_j\setminus\{\omega_{\mathrm{j}}\},F\right)),$$

which finishes the inductive proof. The last assertion follows now from Theorem 3.2.2 a), since $\mathbb{S}_{k_j}\setminus\{\omega_{\mathrm{j}}\}$ is homeomorphic to $\mathbb{R}^{k_j}$. ■

PROPOSITION 3.5.7 *Let Ω_1,Ω_2 be locally compact spaces such that the $E-C*-$algebra $\mathscr{C}_0(\Omega_2,F)$ is K-null, Γ a compact set of Ω_1, and $\vartheta:\Gamma\longrightarrow\Omega_2$ a continuous map. We denote by Ω the locally compact space obtained from the topological sum of Ω_1 and Ω_2 by identifying the points ω and $\vartheta(\omega)$ for all $\omega\in\Gamma$.*

a) If

$$\varphi:\mathscr{C}_0(\Omega_1\setminus\Gamma,F)\longrightarrow\mathscr{C}_0(\Omega,F)$$

denotes the inclusion map then

$$K_i(\varphi):K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma,F))\longrightarrow K_i(\mathscr{C}_0(\Omega,F))$$

is a group isomorphism. If in addition $\Omega\in\Upsilon$ or $\Omega_1\setminus\Gamma\in\Upsilon$ then

$$\Omega,\Omega_1\setminus\Gamma\in\Upsilon,\; p(\Omega)=p(\Omega_1\setminus\Gamma),\; q(\Omega)=q(\Omega_1\setminus\Gamma),\; \Omega_\Upsilon=(\Omega_1\setminus\Gamma)_\Upsilon\,.$$

b) If Ω^* denotes the Alexandroff compactification of Ω then

$$K_i(\mathscr{C}(\Omega^*,F))\approx K_i(F)\times K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma,F))\,.$$

a) If we put

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Omega_2, F), \quad x \longmapsto x|\Omega_2$$

then

$$0 \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0(\Omega_2, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. Since $\mathscr{C}_0(\Omega_2, F)$ is K-null, the assertion follows from the Topological six-term sequence (Proposition 2.1.8 c)).

b) follows from a) and Alexandroff's K-theorem (Theorem 2.2.1 a)). ■

COROLLARY 3.5.8 *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces, $\omega_j \in \Omega_j$ for all $j \in J$, and Ω the locally compact space obtained from the topological sum of the family $(\Omega_j)_{j\in J}$ by identifying the points ω_j for all $j \in J$.*

a) If there is a $j_0 \in J$ such that $\mathscr{C}_0\left(\Omega_{j_0}, F\right)$ is K-null then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx \prod_{j\in J\setminus\{j_0\}} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_j\}, F\right)).$$

b) If $\Omega_j := [0,1[$ for all $j \in J$ and $n := Card\, J$ then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(F)^{n-1}.$$

c) Let $j_0 \in J$ and $\Omega_{j_0} := [0,1[$. If $(k_j)_{j\in J\setminus\{j_0\}}$ is a family in $\mathbb{N}$,

$$p := Card\left\{\, j \in J \setminus \{j_0\} \mid k_j \text{ is even} \right\},$$

$$q := Card\left\{\, j \in J \setminus \{j_0\} \mid k_j \text{ is odd} \right\},$$

and $\Omega_j := \mathbf{S}_{k_j}$ for every $j \in J \setminus j_0$ then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_i(F)^p \times K_{i+1}(F)^q.$$

a) Let Ω' be the locally compact space obtained from the topological sum of the family $(\Omega_j)_{J\setminus\{j_0\}}$ by identifying the points ω_j for all $j \in J \setminus \{j_0\}$ and let $\bar{\omega}$ denote the point obtained by this identification. If we replace in Proposition 3.5.7 Ω_1 by Ω', Ω_2 by Ω_{j_0}, Γ by $\bar{\omega}$, and take $\vartheta(\bar{\omega}) := \omega_{j_0}$ then we get

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_i\left(\mathscr{C}_0\left(\Omega' \setminus \{\bar{\omega}\}, F\right)\right).$$

$\Omega' \setminus \{\bar{\omega}\}$ is the topological sum of the family $(\Omega_j \setminus \{\omega_{\mathrm{j}}\})_{j\in J\setminus\{\mathrm{j}_0\}}$ so by the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left(\Omega' \setminus \{\bar{\omega}\}, F\right)\right) \approx \prod_{j\in J\setminus\{\mathrm{j}_0\}} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_{\mathrm{j}}\}, F\right)) \,.$$

b) follows immediately from a) since $\mathscr{C}_0\left([0,1[, F\right)$ is K-null and

$$K_i(\mathscr{C}_0\left([0,1[\setminus\{\omega\}, F\right)) \approx K_{i+1}(F)$$

for all $\omega \in [0,1[$.

c) For $j \in J \setminus \{\mathrm{j}_0\}$, $\mathbf{S}_{k_j} \setminus \{\omega_{\mathrm{j}}\}$ is homeomorphic to $\mathrm{I\!R}^{k_j}$ and so by Theorem 3.2.2 a), $K_i\left(\mathscr{C}_0\left(\mathbf{S}_{k_j} \setminus \{\omega_{\mathrm{j}}\}, F\right)\right) \approx K_{i+k_j}(F)$. Since $\mathscr{C}_0\left([0,1[, F\right)$ is K-null, we get from a),

$$K_i(\mathscr{C}_0\left(\Omega, F\right)) \approx K_i(F)^p \times K_{i+1}(F)^q \,. \qquad \blacksquare$$

COROLLARY 3.5.9 *Let J_1, J_2, J_3 be pairwise disjoint finite sets and let Ω be the locally compact space (the* **graph***) obtained from the topological sum of* $[0,1] \times J_1$, $[0,1[\times J_2$, *and* $]0,1[\times J_3$ *by identifying some of the points of the set*

$$\{\ (0, j) \mid j \in J_1 \cup J_2\,\} \cup \{\ (1, j) \mid j \in J_1\ \} \,.$$

If s denotes the number of compact connected components of Ω and r_0 and r_1 denote the number of vertices and chords of the graph Ω, respectively, then

$$K_i(\mathscr{C}_0\left(\Omega, F\right)) \approx K_i(F)^s \times K_{i+1}(F)^{s+r_1-r_0} \,.$$

By the Product Theorem (Proposition 2.3.1 a)), we may assume Ω connected.

Assume first there is a $j \in J_3$ such that Ω contains $]0,1[\times\{\mathrm{j}\}$. Since Ω is connected, $\Omega =]0,1[\times\{\mathrm{j}\}$. Thus Ω is homeomorphic to $\mathrm{I\!R}$, $r_1 - r_0 = 1$, and the assertion follows from Theorem 3.2.2 a).

Assume now there is a $j \in J_2$ such that Ω contains $[0,1[\times\{\mathrm{j}\}$. By Proposition 3.5.7 a),

$$K_i(\mathscr{C}_0\left(\Omega, F\right)) \approx K_i(\mathscr{C}_0\left(\Omega \setminus ([0,1[\times\{\mathrm{j}\}), F\right)) \,.$$

Ω and $\Omega \setminus ([0,1[\times\{\mathrm{j}\})$ have the same $r_1 - r_0$, so we may replace Ω by $\Omega \setminus ([0,1[\times\{\mathrm{j}\})$. Repeating the operation, we obtain finally a locally compact space, which is the topological sum of a finite family $(]0,1[)_{j\in J}$, and in this case the assertion follows from the Product Theorem (Proposition 2.3.1 a)) and Theorem 3.2.2 a).

Finally assume Ω compact. Then there is a $j \in J_1$ such that Ω contains $[0,1]\times\{\mathrm{j}\}$. By the above and by Alexandroff's K-theorem

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_i(\mathscr{C}_0(\Omega\setminus\{(1,\mathrm{j})\},F))\ .$$

If s', r'_0, r'_1 denote the corresponding numbers associated to $\Omega\setminus\{(1,\mathrm{j})\}$ then $s'=0$, $r'_0 = r_0-1$, and $r'_1 = r_1$. All the connected components of $\Omega\setminus\{(1,\mathrm{j})\}$ satisfy the condition of the above paragraphs, so

$$K_i(\mathscr{C}_0(\Omega\setminus\{(1,\mathrm{j})\},F)) \approx K_{i+1}(F)^{r'_1-r'_0} \approx K_{i+1}(F)^{1+r_1-r_0}\ ,$$

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_{i+1}(F)^{1+r_1-r_0}\ . \qquad \blacksquare$$

COROLLARY 3.5.10 *If Ω is a compact graph contained in $\mathbb{B}_n$ then*

$$K_i(\mathscr{C}_0(\mathbb{B}_n\setminus\Omega,F)) \approx K_i(F)^{s-r_0+r_1}\times K_{i+1}(F)^{s-1}\ ,$$

where s denotes the number of connected components of Ω and r_0 and r_1 the munber of vertices and chords of Ω, respectively.

Let ω be a vertex of Ω. By Corollary 3.5.9 and Corollary 2.4.4 a),

$$K_i(\mathscr{C}_0(\Omega\setminus\{\omega\},F)) \approx K_i(F)^{s-1}\times K_{i+1}(F)^{s-r_0+r_1}$$

and by Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\mathbb{B}_n\setminus\Omega,F)) \approx K_{i+1}(\mathscr{C}_0(\Omega\setminus\{\omega\},F)) \approx K_i(F)^{s-r_0+r_1}\times K_{i+1}(F)^{s-1}\ . \qquad \blacksquare$$

EXAMPLE 3.5.11 *Let $n\in\mathbb{N}$, Γ a closed set of $\mathbf{S}_n$, $\emptyset\neq\Gamma\neq\mathbf{S}_n$, $\omega\in\Gamma$, Γ' the compact space obtained from $\Gamma\times[0,1]$ by identifying the points of $\Gamma\times 0$, and Ω the compact space obtained from the topological sum of $\mathbf{S}_n$ and Γ' by identifying the points of $\Gamma\subset\mathbf{S}_n$ with the points of $\Gamma\times\{1\}\subset\Gamma'$.*

a) $K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_{i+n}(F)\times K_{i+1}(\mathscr{C}_0(\Gamma\setminus\{\omega\},F))$.

b) If Γ *is finite then*

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F) \times K_{i+n}(F) \times K_{i+1}(F)^{Card\,\Gamma-1} .$$

c) If Γ *is a graph then*

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)^{1+s+r_1-r_0} \times K_{i+n}(F) \times K_{i+1}(F)^{s-1} ,$$

where s *denotes the number of connected components of* Ω *and* r_0 *and* r_1 *denote the number of vertices and chords of the graph* Γ*, respectively,*

a) By Theorem 3.2.2 e_1),

$$K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) .$$

By Proposition 2.4.1, $\mathscr{C}_0(\Gamma' \setminus \{0\}, F)$ is K-null, where 0 is the point obtained from the identification of the points of $\Gamma \times \{0\}$. By Proposition 3.5.7 a),

$$K_i(\mathscr{C}_0(\Omega \setminus \{0\}, F)) \approx K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \Gamma, F)) ,$$

so by Alexandroff's K-theorem (Theorem 2.2.1 a)),

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F) \times K_{i+n}(F) \times K_{i+1}(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) .$$

b) follows from a) and the Product Theorem (Proposition 2.3.1 a)).

c) By Corollary 3.5.9 and Alexandroff's K-theorem (Theorem 2.2.1 a)),

$$K_i(\mathscr{C}(\Gamma,F)) \approx K_i(F)^s \times K_{i+1}(F)^{s+r_1-r_0} ,$$

$$K_i(\mathscr{C}_0(\Gamma \setminus \{\omega\},F)) \approx K_i(F)^{s-1} \times K_{i+1}(F)^{s+r_1-r_0} ,$$

so by a),

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)^{1+s+r_1-r_0} \times K_{i+n}(F) \times K_{i+1}(F)^{s-1} .$$ ■

PROPOSITION 3.5.12 *Let* $(p_j)_{j\in J}$ *be a finite family in* $\mathbb{N}$*,* $(J \neq \emptyset)$*, and for every* $j \in J$ *put* $\Omega_j := \mathbf{S}_{p_j}$*. Let* Ω' *be the topological sum of the family* $(\Omega_j)_{j\in J}$*,* $(\Gamma_k)_{k\in K}$ *a finite family of pairwise disjoint nonempty finite subsets of* Ω'*,* $\Gamma := \bigcup\limits_{k\in K} \Gamma_k$*, and* Ω *the compact space obtained from* Ω' *by identifying for every* $k \in K$ *the points of* Γ_k*. If* Ω *is connected then*

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F) \times K_{i+1}(F)^{Card\,\Gamma - Card\,J - Card\,K + 1} \times \prod_{j\in J} K_{i+p_j}(F) .$$

If $K=\emptyset$, since Ω is connected, J is a one-point set and the assertion holds by Theorem 3.2.2 b). Thus we may assume $K=\mathbb{N}_{\mathrm{n}}$ for some $n\in\mathbb{N}$. Take $k_1\in K$ and put $J_1:=\left\{\, j\in J \mid \Omega_j\cap\Gamma_{k_1}\neq\emptyset \,\right\}$. We define recursively an injective family $(k_m)_{m\in\mathbb{N}_{\mathrm{n}}}$ in K and an increasing family $(J_m)_{m\in\mathbb{N}_{\mathrm{n}}}$ of subsets of J in the following way. Let $m\in\mathbb{N}_{\mathrm{n}}$, $m>1$, and assume the families were defined up to $m-1$. Since Ω is connected there is a $k_m\in K\setminus\left\{\, k_q \;\middle|\; q\in\mathbb{N}_{\mathrm{m-1}} \,\right\}$ such that $\Gamma_{k_m}\cap J_{m-1}\neq\emptyset$. We put

$$J_m:=\left\{\, j\in J \mid \Omega_j\cap\left(\bigcup_{q=1}^{m}\Gamma_{k_q}\right)\neq\emptyset \,\right\} .$$

It is easy to prove by induction with respect to $m\in\mathbb{N}_{\mathrm{n}}$ that

$$Card\left(\bigcup_{q=1}^{m}\Gamma_{k_q}\right)-Card\,J_m-m+1\geq 0$$

for every $m\in\mathbb{N}_{\mathrm{n}}$. In particular,

$$Card\,\Gamma-Card\,J-Card\,K+1\geq 0\,.$$

For every $j\in J$, by Proposition 2.4.11 and Theorem 3.2.2 a),

$$K_i\left(\mathscr{C}_0\left(\Omega_j\setminus\Gamma,F\right)\right)\approx K_{i+1}\left(F\right)^{Card\,(\Gamma\cap\Omega_j)-1}\times K_{i+p_j}\left(F\right)$$

so that by the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left(\Omega'\setminus\Gamma,F\right)\right)\approx K_{i+1}\left(F\right)^{Card\,\Gamma-Card\,J}\times\prod_{j\in J}K_{i+p_j}\left(F\right)\,.$$

For every $k\in K$ let ω_k be the point of Ω corresponding to the unified points of Γ_k and put $\Delta:=\{\,\omega_k \mid k\in K\,\}$. Then by Proposition 2.4.11,

$$K_i\left(\mathscr{C}_0\left(\Omega\setminus\Delta,F\right)\right)\approx K_i\left(\mathscr{C}_0\left(\Omega\setminus\{\omega_{\mathrm{k}_0}\},F\right)\right)\times K_{i+1}\left(F\right)^{Card\,K-1}\,,$$

where $k_0\in K$. By the above and by Alexandroff's K-theorem, since $\Omega\setminus\Delta=\Omega'\setminus\Gamma$,

$$K_i\left(\mathscr{C}\left(\Omega,F\right)\right)\times K_{i+1}\left(F\right)^{Card\,K-1}\approx$$

$$\approx K_i\left(F\right)\times K_i\left(\mathscr{C}_0\left(\Omega\setminus\{\omega_{\mathrm{k}_0}\},F\right)\right)\times K_{i+1}\left(F\right)^{Card\,K-1}\approx$$

$$\approx K_i\left(F\right)\times K_i\left(\mathscr{C}_0\left(\Omega\setminus\Delta,F\right)\right)\approx K_i\left(F\right)\times K_i\left(\mathscr{C}_0\left(\Omega'\setminus\Gamma,F\right)\right)\approx$$

$$\approx K_i\left(F\right)\times K_{i+1}\left(F\right)^{Card\,\Gamma-Card\,J-Card\,K+1}\times K_{i+1}\left(F\right)^{Card\,K-1}\times\prod_{j\in J}K_{i+p_j}\left(F\right)\,,$$

$$K_i\left(\mathscr{C}\left(\Omega,F\right)\right)\approx K_i\left(F\right)\times K_{i+1}\left(F\right)^{Card\,\Gamma-Card\,J-Card\,K+1}\times\prod_{j\in J}K_{i+p_j}\left(F\right)\,.$$ ■

COROLLARY 3.5.13 *Let $(p_j)_{j\in\mathbb{N}_n}$ be a family in $\mathbb{N}$ and for every $j\in\mathbb{N}_n$ put $\Omega_j := \mathbb{S}_{p_j}$. For every $j\in\mathbb{N}_n$ let Γ_j and Γ'_j be disjoint nonempty finite subsets of Ω_j such that $k_j := Card\,\Gamma'_j = Card\,\Gamma_{j+1}$ for every $j\in\mathbb{N}_{n-1}$. We denote by Ω the compact space obtained from the topological sum of the family $(\Omega_j)_{j\in\mathbb{N}_n}$ by identifying in a bijective way Γ'_j with Γ_{j+1} for all $j\in\mathbb{N}_{n-1}$. Then*

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_{i+1}(F)^{\sum\limits_{j=1}^{n-1}(k_j-1)} \times \prod_{j=1}^{n} K_{i+p_j}(F)\,. \qquad \blacksquare$$

PROPOSITION 3.5.14 *Let Ω_1,Ω_2 be locally compact spaces and for every $j\in\{1,2\}$ let Γ_j be a compact set of Ω_j and $\vartheta_j : \mathbb{B}_n \longrightarrow \Gamma_j$ a homeomorphism such that $\Delta_j := \vartheta_j(\mathbb{B}_n\setminus\mathbb{S}_{n-1})$ is an open set of Ω_j. We denote by Ω the locally compact space obtained from the topological sum of $\Omega_1\setminus\Delta_1$ and $\Omega_2\setminus\Delta_2$ by identifying the points $\vartheta_1(\omega)$ and $\vartheta_2(\omega)$ for all $\omega\in\mathbb{S}_{n-1}$. Then for every $\omega\in\mathbb{S}_{n-1}$,*

$$K_i(\mathscr{C}_0(\Omega\setminus\{\vartheta_1(\omega)\},F)) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma_1,F))\times K_i(\mathscr{C}_0(\Omega_2\setminus\Gamma_2,F))\times K_{i+n-1}(F)\ .$$

We use the notation of the topological triple (Proposition 2.1.11), which we mark with a prime in order to distinguish them from the present notation. We put $\Omega'_2 := \Omega\setminus\{\vartheta_1(\omega)\}$ and take as Ω'_3 the topological sum of $\Omega_1\setminus\Gamma_1$ and $\Omega_2\setminus\Gamma_2$ and as Ω'_1 the locally compact space obtained from Ω by completing first $\vartheta_1(\mathbb{S}_{n-1})$ to $\vartheta_1(\mathbb{B}_n)$ and deleting then ω. By the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left(\Omega'_3,F\right)\right) \approx K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma_1,F))\times K_i(\mathscr{C}_0(\Omega_2\setminus\Gamma_2,F))\ .$$

Since $\Omega'_2\setminus\Omega'_3$ is homeomorphic to $\mathbb{S}_{n-1}\setminus\{\omega\}$, we get by Theorem 3.2.2 e_1),

$$K_i\left(\mathscr{C}_0\left(\Omega'_2\setminus\Omega'_3,F\right)\right) \approx K_{i+n-1}(F)\ .$$

Thus by the topological triple (Proposition 2.1.11 b_3)) (and Theorem 3.1.2 b)),

$$K_i(\mathscr{C}_0(\Omega\setminus\{\vartheta_1(\omega)\},F)) \approx K_i\left(\mathscr{C}_0\left(\Omega'_2,F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega'_3,F\right)\right)\times K_i\left(\mathscr{C}_0\left(\Omega'_2\setminus\Omega'_3,F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma_1,F))\times K_i(\mathscr{C}_0(\Omega_2\setminus\Gamma_2,F))\times K_{i+n-1}(F)\ . \qquad \blacksquare$$

COROLLARY 3.5.15 *If S_g is an orientable compact connected surface of genus $g \in \mathbb{N}$ and Γ is a nonempty finite subset of S_g then*

$$K_i(\mathscr{C}(S_g,F)) \approx K_i(F)^{g+1} \times K_{i+1}(F)^{3g-1}\ ,$$

$$K_i(\mathscr{C}(S_g \setminus \Gamma,F)) \approx K_i(F)^{g} \times K_{i+1}(F)^{3g-2+Card\,\Gamma}\ .$$

Assume first Γ is a one-point set $\{\omega\}$. We prove the second assertion in this case by induction with respect to $g \in \mathbb{N}$. By Proposition 3.2.15 b), the assertion holds for $g = 1$. Assume now the assertion holds for $g \in \mathbb{N}$. Let Δ_1 be a closed disc of S_1, Δ_g a closed disc of S_g, $\omega \in \Delta_1$, and $\omega \in \Delta_g$. $S_{g+1} \setminus \{\omega\}$ can be obtained from the topological sum of $S_1 \setminus \Delta_1$, $S_g \setminus \Delta_2$, and $\mathbf{S}_1 \setminus \{\omega\}$ by pasting $\mathbf{S}_1 \setminus \{\omega\}$ in the the boundaries of $\Delta_1 \setminus \{\omega\}$ and $\Delta_g \setminus \{\omega\}$. By the induction hypothesis, since $S_g \setminus \Delta_g$ is homeomorphic to $S_g \setminus \{\omega\}$,

$$K_i(\mathscr{C}_0(S_g \setminus \Delta_g,F)) \approx K_i(F)^{g} \times K_{i+1}(F)^{3g-1}\ .$$

By Proposition 3.5.14,

$$K_i(\mathscr{C}_0(S_{g+1} \setminus \{\omega\},F)) \approx K_i(F)^{g+1} \times K_{i+1}(F)^{3g+2}\ ,$$

which finishes the inductive proof.

The first assertion follows now from Alexandroff's K-theorem (Proposition 2.2.1 a)) and the second one from Proposition 2.4.11. ■

The following Example shows a way to generalize Corollary 3.5.15.

EXAMPLE 3.5.16 *Let Ω be the compact space obtained from the topological sum of $\mathbf{S}_1 \times \mathbf{S}_2 \setminus \Delta$, $\mathbf{S}_1 \times \mathbf{S}_1 \times \mathbf{S}_1 \setminus \Delta'$, and $\mathbf{S}_2$, where Δ and Δ' denote balls homeomorphic to $\mathbb{B}_3$ by pasting $\mathbf{S}_2$ in the boundaries of Δ and Δ'. Then for every nonempty finite subset Γ of Ω,*

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)^{5} \times K_{i+1}(F)^{6}\ ,$$

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma,F)) \approx K_i(F)^{4} \times K_{i+1}(F)^{5+Card\,\Gamma}\ .$$ ■

Remark. Let

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow{\psi} F_3 \longrightarrow 0\,,$$

$$0 \longrightarrow G_1 \xrightarrow{\varphi'} G_2 \xrightarrow{\psi'} G_3 \longrightarrow 0$$

be exact sequences in $\mathfrak{M}_E$ and $\lambda : F_3 \longrightarrow G_3$ and isomorphism in $\mathfrak{M}_E$. Then

$$H := \left\{ (x,y) \in F_2 \times G_2 \mid \psi' y = \lambda \psi x \right\}$$

is a C*-subalgebra of $F_2 \times G_2$ containing the ideal $F_1 \times G_1$ of $F_2 \times G_2$. H corresponds to the operation of pasting F_2 and G_2 in $\mathfrak{M}_E$.

Chapter 4

Some Supplementary Results

Throughout this chapter F denotes an E-C*-algebra.

4.1 Full E-C*-algebras

DEFINITION 4.1.1 *A* **full E-C*-algebra** *is a unital C*-algebra F for which E is a* canonical *unital C*-subalgebra such that $\alpha x = x\alpha$ for all $(\alpha, x) \in E \times F$. Every full E-C*-algebra is canonically an E-C*-algebra, the exterior multiplication being the restriction of the interior multiplication. We denote by $\mathfrak{C}_E$ the category of full E-C*-algebras for which the morphisms are the unital E-linear C*-homomorphisms. In particular $\mathfrak{C}_{\mathbb{C}}$ is the category of all unital C*-algebras with unital C*-homomorphisms. A* **full E-C*-subalgebra of** *F is a C*-subalgebra of F containing E. An isomorphism of full E-C*-algebras is also called* **E-C*-isomorphism**.

If $\prod_{j\in J} F_j$ is a finite family of full E-C-algebras, $J \neq \emptyset$, then $\prod_{j\in J} F_j$ is a full E-C*-algebra, the canonical embedding $E \to \prod_{j\in J} F_j$ being given by*

$$E \longrightarrow \prod_{j\in J} F_j, \quad \alpha \longmapsto (\alpha)_{j\in J} .$$

If F is a full E-C*-algebra and G a unital C*-algebra then the map

$$E \longrightarrow F \otimes G, \quad \alpha \longmapsto \alpha \otimes 1_G$$

is an injective C*-homomorphism. In particular, the E-C*-algebra $F \otimes G$ has a canonical structure of a full E-C*-algebra.

PROPOSITION 4.1.2 *Let F be an E-C*-algebra. We denote by $\check{F}$ the vector space $E \times F$ endowed with the bilinear map*

$$(E \times F) \times (E \times F) \longrightarrow E \times F, \quad ((\alpha, x), (\beta, y)) \longmapsto (\alpha\beta, \alpha y + \beta x + xy)$$

and with the involution

$$E \times F \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha^*, x^*) .$$

a) $\check{F}$ is an involutive unital algebra with $(1_E, 0)$ as unit and $\{\, (\alpha, 0) \mid \alpha \in E \,\}$ is a unital involutive subalgebra of $\check{F}$ isomorphic to E.

b) If E and F are C-subalgebras of a C*-algebra G then the map*

$$\varphi : \check{F} \longrightarrow E \times G\,, \quad (\alpha,x) \longmapsto (\alpha, \alpha + x)$$

is an injective involutive algebra homomorphism with closed image

$$\{\,(\alpha,y) \in E \times G \mid \alpha - y \in F\,\}\ .$$

In particular $\varphi(\check{F})$ is a C-subalgebra of $E \times G$ and there is a norm on $\check{F}$ with respect to which $\check{F}$ is a C*-algebra.*

c) There is a unique C-norm on $\check{F}$ making it a C*-algebra. Moreover $\check{F}$ is a full E-C*-algebra and F may be identified with the closed ideal*

$$\{\,(0,x) \mid x \in F\,\}$$

of $\check{F}$. We shall always consider $\check{F}$ endowed with the structure of a full E-C-algebra.*

d) If F is a full E-C-algebra then the map*

$$\check{F} \longrightarrow E \times F, \quad (\alpha,x) \longmapsto (\alpha, \alpha + x)$$

is an isomorphism of E-C-algebras with inverse*

$$E \times F \longrightarrow \check{F}, \quad (\alpha,x) \longmapsto (\alpha, x - \alpha)\ .$$

e) If $E = \mathbb{C}$ then $\check{F}$ is the unitization $\tilde{F}$ of F.

a) is easy to verify.

b) Only the assertion that the image of φ is closed needs a proof. Let $(\alpha,x) \in \overline{\varphi(\check{F})}$. There are sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ in E and F, respectively, such that

$$\lim_{n\to\infty} (\alpha_n, \alpha_n + x_n) = (\alpha,x)\ .$$

It follows

$$\alpha = \lim_{n\to\infty} \alpha_n \in E\,, \quad x - \alpha = \lim_{n\to\infty} x_n \in F\,, \quad (\alpha,x) = \varphi(\alpha, x-\alpha) \in \varphi(\check{F})\ .$$

Thus $\varphi(\check{F})$ is closed.

c) Let Ω be the spectrum of E and $\tilde{F}$ the unitization of F. Then E and F are C*-subalgebras of the C*-algebra $\mathscr{C}\left(\Omega,\tilde{F}\right)$ and the assertion follows from b).

d) follows from c) and b).

e) is obvious. ∎

EXAMPLE 4.1.3 *Let F be a commutative E-C*-algebra.*

a) $\check{F}$ is commutative. We denote by Ω_E, Ω_F, and $\Omega_{\check{F}}$ the spectra of E, F, and $\check{F}$, respectively.

b) Ω_F is homeomorphic to an open set Ω' of $\Omega_{\check{F}}$ such that $F \approx \mathscr{C}_0(\Omega', \mathbf{C})$.

c) There is a unique surjective continuous map $\vartheta : \Omega_{\check{F}} \longrightarrow \Omega_E$ such that if we put

$$\phi : E \approx \mathscr{C}(\Omega_E, \mathbf{C}) \longrightarrow \check{F} \approx \mathscr{C}(\Omega_{\check{F}}, \mathbf{C})\,, \quad \alpha \longmapsto \alpha \circ \vartheta$$

then ϕ is an injective continuous C-homomorphism (so we may identify E with $\phi(E)$).*

d) The restriction of ϑ to $\Omega_{\check{F}} \setminus \Omega'$ is a homeomorphism.

e) If F is unital then $\Omega_{\check{F}}$ is homeomorphic to the topological sum of Ω_E and Ω_F.

a) is easy to see.

b) follows from the fact that F may be identified with a closed ideal of $\check{F}$ (Proposition 4.1.2 c)).

c) is proved in [1] Proposition 4.1.2.15.

d) Let $\omega \in \Omega_E$ and put

$$\omega' : \check{F} \longrightarrow \mathbf{C}, \quad (\alpha, x) \longmapsto \alpha(\omega)\,.$$

Then $\omega' \in \Omega_{\check{F}} \setminus \Omega'$ and $\vartheta(\omega') = \omega$, so $\vartheta|(\Omega_{\check{F}} \setminus \Omega')$ is surjective.

Let $\omega_1, \omega_2 \in \Omega_{\check{F}} \setminus \Omega'$, $\omega_1 \neq \omega_2$. There is an $(\alpha, x) \in \check{F}$ with

$$\langle(\alpha, x), \omega_1\rangle \neq \langle(\alpha, x), \omega_2\rangle\,.$$

Since $\langle(\alpha,x),\omega_j\rangle = \langle\alpha,\omega_j\rangle$ for every $j\in\{1,2\}$, $\vartheta|(\Omega_{\check{F}}\setminus\Omega')$ is injective.

e) follows from d) since in this case Ω' is clopen. ∎

Remark. The above d) may be seen as a kind of generalization of Alexandroff's compactification.

DEFINITION 4.1.4 *We put for every E-C*-algebra F*

$$\iota^F : F \longrightarrow \check{F}\,, \quad x \longmapsto (0,x)\,,$$

$$\pi^F : \check{F} \longrightarrow E\,, \quad (\alpha,x) \longmapsto \alpha\,,$$

$$\lambda^F : E \longrightarrow \check{F}\,, \quad \alpha \longmapsto (\alpha,0)\,,$$

$$\sigma^F := \lambda^F \circ \pi^F\,.$$

If $E=\mathbb{C}$ then

$$\check{F}=\tilde{F}\,, \qquad \iota_F=\iota^F\,, \qquad \pi_F=\pi^F\,, \qquad \lambda_F=\lambda^F\,.$$

All these maps are E-linear C*-homomorphisms,

$$\pi^F\circ\iota^F=0, \qquad \pi^F\circ\lambda^F=id_E, \qquad \pi^F\circ\sigma^F=\pi^F\,,$$

ι^F and λ^F are injective, π^F, λ^F, and σ^F are unital, and

$$0\longrightarrow F\xrightarrow{\iota^F}\check{F}\underset{\overleftarrow{\lambda^F}}{\overset{\pi^F}{\longrightarrow}}E\longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$.

PROPOSITION 4.1.5

a) If $F\overset{\varphi}{\to}F'$ is a morphism in $\mathfrak{M}_E$ then the map

$$\check{\varphi}:\check{F}\longrightarrow\check{F}'\,, \quad (\alpha,x)\longmapsto(\alpha,\varphi x)$$

is an involutive unital algebra homomorphism, injective or surjective if φ is so. If $F=F'$ and if φ is the identity map then $\check{\varphi}$ is also the identity map.

b) Let F_1, F_2, F_3 be E-C-algebras and let $\varphi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ be E-linear C*-homomorphisms. Then* $\overbrace{\psi \circ \varphi}^{\vee} = \check{\psi} \circ \check{\varphi}$. ■

Remark. If $E = \mathbb{C}$ then $\check{\varphi} = \tilde{\varphi}$.

EXAMPLE 4.1.6 *Let F be a full E-C*-algebra and F' a closed ideal of F.*

a) F' endowed with the exterior multiplication

$$E \times F' \longrightarrow F', \quad (\alpha, x) \longmapsto \alpha x$$

is an E-C-algebra.*

b) The map

$$\check{F'} \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an injective E-linear C-homomorphism with image*

$$\left\{ (\alpha, x) \in E \times F \mid \alpha - x \in F' \right\} .$$

c) $\mathfrak{C}_E$ is a full subcategory of $\mathfrak{M}_E$. ■

PROPOSITION 4.1.7 *Let F be a full E-C*-algebra and J a finite set.*

a) $F^J = F \otimes l^2(J)$ endowed with the maps

$$F \times F^J \longrightarrow F^J, \quad (x, \xi) \longmapsto (x\xi_j)_{j \in J},$$

$$F^J \times F \longrightarrow F^J, \quad (\xi, x) \longmapsto (\xi_j x)_{j \in J},$$

$$F^J \times F^J \longrightarrow F, \quad (\xi, \eta) \longmapsto \sum_{j \in J} \eta_j^* \xi_j$$

is a unital Hilbert F-module ([1] Proposition 5.6.4.2 c)).

b) Let $\mathscr{L}(F^J)$ be the Banach space of operators on F^J. The set $\mathscr{L}_F(F^J)$ of adjointable operators on F^J is a Banach subspace of $\mathscr{L}(F^J)$. $\mathscr{L}_F(F^J)$ endowed with the restriction of the norm of $\mathscr{L}(F^J)$ it is a full E-C-algebra* ([1] Theorem 5.6.1.11 d), [1] Proposition 5.6.1.8 g),h)). ■

PROPOSITION 4.1.8 *For every E-C*-algebra F the sequence*

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\iota^F)} K_i(\check{F}) \underset{\xleftarrow[K_i(\lambda^F)]{}}{\xrightarrow{K_i(\pi^F)}} K_i(E) \longrightarrow 0$$

is split exact and the map

$$K_i(F) \times K_i(E) \longrightarrow K_i(\check{F}), \quad (a,b) \longmapsto K_i(\iota^F)\,a + K_i(\lambda^F)\,b$$

is a group isomorphism.

Since the sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\xleftarrow[\lambda^F]{}}{\xrightarrow{\pi^F}} E \longrightarrow 0$$

is split exact the assertion follows from the split exact axiom (Axiom 1.2.3). ■

COROLLARY 4.1.9 *Let G be a C*-algebra.*

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G \xrightarrow{\iota^F \otimes id_G} \check{F} \otimes G \underset{\xleftarrow[\lambda^F \otimes id_G]{}}{\xrightarrow{\pi^F \otimes id_G}} E \otimes G \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(\iota^F \otimes id_G)} K_i(\check{F} \otimes G) \underset{\xleftarrow[K_i(\lambda^F \otimes id_G)]{}}{\xrightarrow{K_i(\pi^F \otimes id_G)}} K_i(E \otimes G) \longrightarrow 0$$

is split exact and the map

$$K_i(E \otimes G) \times K_i(F \otimes G) \longrightarrow K_i(\check{F} \otimes G),$$

$$(a,b) \longmapsto K_i(\lambda^F \otimes id_G)\,a + K_i(\iota^F \otimes id_G)\,b$$

is a group isomorphism.

c) Let $F \xrightarrow{\varphi} F'$ *be a morphism in* $\mathfrak{M}_E$ *and* $G \xrightarrow{\psi} G'$ *a morphism in* $\mathfrak{M}_{\mathbf{C}}$*. If we identify the isomorphic groups of b) then*

$$K_i(\check{\varphi} \otimes \psi) : K_i(\check{F} \otimes G) \longrightarrow K_i(\check{F}' \otimes G'),$$

$$(a,b) \longmapsto (K_i(id_E \otimes \psi)\,a, K_i(\varphi \otimes \psi)\,b)$$

is a group isomorphism.

a) follows from Proposition 1.4.8 a).

b) follows from a) and the split exact axiom (Axiom 1.2.3).

c) follows from b) and the commutativity of the following diagram:

$$\begin{tikzcd} F\otimes G \arrow[r, "\iota^F\otimes id_G"] \arrow[d, "\varphi\otimes\psi"'] & \check{F}\otimes G \arrow[d, "\check{\varphi}\otimes\psi"'] & E\otimes G \arrow[l, "\lambda^F\otimes id_G"'] \arrow[d, "id_E\otimes\psi"] \\ F'\otimes G' \arrow[r, "\iota^{F'}\otimes id_{G'}"'] & \check{F}'\otimes G' & E\otimes G' \arrow[l, "\lambda^{F'}\otimes id_{G'}"] \end{tikzcd} .$$ ■

COROLLARY 4.1.10 *Let $F \overset{\phi_1}{\to} F'$ and $F \overset{\phi_2}{\to} F'$ be morphisms in $\mathfrak{M}_E$. If F is K-null then $K_i\left(\check{\phi}_1\right) = K_i\left(\check{\phi}_2\right)$.*

By Proposition 4.1.8, the map

$$K_i(F)\times K_i(E) \longrightarrow K_i\left(\check{F}\right), \quad (a,b) \longmapsto K_i\left(\iota^F\right)a + K_i\left(\lambda^F\right)b$$

is a group isomorphism. Since F is K-null, $K_i\left(\lambda^F\right)$ is a group isomorphism. We get from $\check{\phi}_1\circ\lambda^F = \check{\phi}_2\circ\lambda^F$,

$$K_i\left(\check{\phi}_1\right)\circ K_i\left(\lambda^F\right) = K_i\left(\check{\phi}_2\right)\circ K_i\left(\lambda^F\right), \qquad K_i\left(\check{\phi}_1\right) = K_i\left(\check{\phi}_2\right).$$ ■

4.2 Continuity and Stability

AXIOM 4.2.1 (Continuity axiom) *If $\{(F_j)_{j\in J}, (\varphi_{j,k})_{j,k\in I}\}$ is an inductive system in $\mathfrak{M}_E$ such that $\varphi_{j,k}$ are injective for all $j,k\in J$, $k<j$, and if $\{F, (\varphi_j)_{j\in J}\}$ denotes its inductive limit in $\mathfrak{M}_E$ then $\{K_i(F), (K_i(\varphi_j))_{j\in J}\}$ is the inductive limit of the inductive system $\{(K_i(F_j))_{j\in J}, (K_i(\varphi_{j,k}))_{j,k\in J}\}$.*

PROPOSITION 4.2.2 *If Ω is a totally disconnected compact space then*

$$K_i(\mathscr{C}(\Omega,F)) \approx \left\{ a\in K_i(F)^\Omega \;\middle|\; a(\Omega) \text{ is finite} \right\}.$$

Let Ξ be the set of clopen partitions of Ω ordered by fineness and for every $\Theta := (\Omega_j)_{j\in J} \in \Xi$ and $x \in F^\Theta$ put

$$\tilde{x} : \Omega \longrightarrow F\,, \quad \omega \longmapsto x(j) \quad \text{for} \quad \omega \in \Omega_j\,.$$

Then the map

$$F^\Theta \longrightarrow \mathscr{C}(\Omega, F), \quad x \longmapsto \tilde{x}$$

is an injective E-C*-homomorphism for every $\Theta \in \Xi$ and $\mathscr{C}(\Omega, F)$ is isomorphic to the corresponding inductive limit in $\mathfrak{M}_E$ of $(F^\Theta)_{\Theta\in\Xi}$. By Lemma 2.1.4 c), $K_i(F^\Theta) \approx K_i(F)^\Theta$ for every $\Theta \in \Xi$ and the assertion follows from the continuity axiom (Axiom 4.2.1). ■

PROPOSITION 4.2.3 *Let ξ be an ordinal number, $(\Omega_\eta)_{\eta<\xi}$ a family of path connected, non-compact, locally compact spaces, and $\omega_\eta \in \Omega_\eta$ for every $\eta < \xi$. We denote by Ω^ξ the locally compact space obtained by endowing the disjoint union of the family of sets $(\Omega_\eta)_{\eta<\xi}$ with the topology for which a subset U of Ω^ξ is open if it has the following properties:*

1) $\Omega_\eta \cap U$ is open for every $\eta < \xi$.

2) If $\omega_\eta \in U$ for some $\eta < \xi$ and if there is a $\zeta < \eta$ with $\eta = \zeta + 1$ then $\Omega_\zeta \setminus U$ is compact.

3) If $\omega_\eta \in U$ for some limit ordinal number $\eta < \xi$ then there is a $\zeta < \eta$ such that $\bigcup_{\zeta<\zeta'<\eta} \Omega_{\zeta'} \subset U$.

If $K_i(\mathscr{C}_0(\Omega_\eta, F)) = 0$ for all $\eta < \xi$ then $K_i\left(\mathscr{C}_0\left(\Omega^\xi, F\right)\right) = 0$.

The assertion is trivial for $\xi = 0$. We prove the general case by transfinite induction. If $\xi = \eta + 1$ for some $\eta < \xi$ for which the assertion holds then by Corollary 3.5.4, the assertion holds also for ξ. If ξ is a limit ordinal number and the assertion holds for every $\eta < \xi$ then by the continuity axiom (Axiom 4.2.1) the assertion holds also for ξ since $\mathscr{C}_0\left(\Omega^\xi, F\right)$ is the inductive limit of the inductive system $\{\, \mathscr{C}_0(\Omega^\eta, F) \mid \eta < \xi \,\}$. ■

Remark. If $\Omega_\eta = [0, 1[$ for every $\eta < \xi$ then Ω^ξ is "one-dimensional".

LEMMA 4.2.4 *Let $\{(F_j)_{j\in J},(\varphi_{j,k})_{j,k\in J}\}$ be an inductive system in $\mathfrak{M}_E$, $\{F,(\varphi_j)_{j\in J}\}$ its inductive limit in $\mathfrak{M}_E$, G an E-C*-algebra, and for every $j\in J$ an injective morphism $\psi_j : F_j \longrightarrow G$ in $\mathfrak{M}_E$ such that $\psi_j = \psi_k \circ \varphi_{k,j}$ for all $j,k\in J$, $j<k$. Then the morphism $\psi : F \longrightarrow G$ in $\mathfrak{M}_E$ such that $\psi_j = \psi\circ\varphi_j$ for all $j,k\in J$, $j<k$,* ([5] Theorem L.2.1) *is injective.*

For $j\in J$ and $x\in F_j$,

$$\left\|\varphi_j x\right\| \le \|x\| = \left\|\psi_j x\right\| = \left\|\psi\varphi_j x\right\| \le \left\|\varphi_j x\right\|,$$

so ψ preserves the norms on $\varphi_j(F_j)$. Since $\bigcup\limits_{j\in J}\varphi_j(F_j)$ is dense in F, ψ preserves the norms, i.e. it is injective. ■

PROPOSITION 4.2.5 *Let $\{(G_j)_{j\in J},(\varphi_{j,k})_{j,k\in J}\}$ be an inductive system in $\mathfrak{M}_{\mathbf{C}}$ such that $\varphi_{j,k}$ are injective for all $j,k\in J$, $k<j$, and let $\{G,(\varphi_j)_{j\in J}\}$ be its inductive limit in $\mathfrak{M}_{\mathbf{C}}$. If $\{F',(\psi_j)_{j\in J}\}$ denotes the inductive limit in $\mathfrak{M}_E$ of the inductive system $\{(F\otimes G_j)_{j\in J},(id_F\otimes\varphi_{j,k})_{j,k\in J}\}$ in $\mathfrak{M}_E$ and $\psi : F' \longrightarrow F\otimes G$ denotes the morphism in $\mathfrak{M}_E$ such that $\psi\circ\psi_j = id_F\otimes\varphi_j$ for all $j\in J$* ([5] Theorem L.2.1) *then ψ is an isomorphism.*

By [5] Corollary T.5.19, $id_F\otimes\varphi_j$ are injective for all $j\in J$. By Lemma 4.2.4, ψ is injective. Since

$$F\otimes\left(\bigcup_{j\in J}G_j\right)\subset Im\,\psi,$$

ψ is surjective and so it is an isomorphism. ■

COROLLARY 4.2.6 *If $\{(G_j)_{j\in J},(\varphi_{j,k})_{j,k\in J}\}$ is an inductive system in $\mathfrak{M}_{\mathbf{C}}$ such that $\varphi_{j,k}$ are injective for all $j,k\in J$, $k<j$, and if $\{G,(\varphi_j)_{j\in J}\}$ is its inductive limit in $\mathfrak{M}_{\mathbf{C}}$ then $\{K_i(F\otimes G),(K_i(id_F\otimes\varphi_j))_{j\in J}\}$ is the inductive limit of the inductive system $\{(K_i(F\otimes G_j))_{j\in J},(K_i\left(id_F\otimes\varphi_{j,k}\right))_{j,k\in J}\}$. In particular if G_j is Υ-null for every $j\in J$ then G is also Υ-null.*

By [5] Corollary T.5.19, $id_F\otimes\varphi_{j,k}$ are injective for all $j,k\in J$, $k<j$. By Proposition 4.2.5, $\{F\otimes G,(id_F\otimes\varphi_j)_{j\in J}\}$ may be identified with the inductive limit in $\mathfrak{M}_E$ of the inductive system $\{(F\otimes G_j)_{j\in J},(id_F\otimes\varphi_{j,k})_{j,k\in J}\}$ in $\mathfrak{M}_E$ and the assertion follows from the continuity axiom (Axiom 4.2.1). ■

COROLLARY 4.2.7 *Let $(G_j)_{j\in J}$ be an infinite family in Υ_1, $\mathfrak{J}$ the set of nonempty finite subsets of J ordered by inclusion, and for all $K,L\in\mathfrak{J}$, $K\subset L$, put $G_K := \bigotimes_{j\in K} G_j$ and*

$$\varphi(L,K) : G_K \longrightarrow G_L\,, \quad \bigotimes_{j\in K} x_j \longmapsto \bigotimes_{j\in L} y_j\,,$$

where

$$y_j := \begin{cases} x_j & \text{if} \quad j\in K \\ 1_{G_j} & \text{if} \quad j\in L\setminus K \end{cases}.$$

Then $\{(G_K)_{K\in\mathfrak{J}}, (\varphi(L,K))_{K,L\in\mathfrak{J}}\}$ is an inductive system in $\mathfrak{M}_{\mathbf{C}}$ and its limit belongs to Υ_1.

We denote by $\{G, (\varphi(K))_{K\in\mathfrak{J}}\}$ the above inductive limit. By Proposition 1.6.5, $G_K\in\Upsilon_1$ for all $K\in\mathfrak{J}$ so by Corollary 4.2.6, $p(G)=1$, $q(G)=0$. Let $F\overset{\phi}{\to}F'$ be a morphism in $\mathfrak{M}_E$ and let $K\in\mathfrak{J}$. Then the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\phi_{G_K,F}} & F\otimes G_K & \xrightarrow{id_F\otimes\varphi(K)} & F\otimes G \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi\otimes id_{G_K}} & & \big\downarrow{\scriptstyle\phi\otimes id_G} \\ F' & \xrightarrow[\phi_{G_K,F'}]{} & F'\otimes G_K & \xrightarrow[id_{F'}\otimes\varphi(K)]{} & F'\otimes G \end{array}$$

is commutative. Since

$$\phi_{G,F} = (id_F\otimes\varphi(K))\circ\phi_{G_K,F}\,, \qquad \phi_{G,F'} = (id_{F'}\otimes\varphi(K))\circ\phi_{G_K,F'}\,,$$

the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\phi_{G,F}} & F\otimes G \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi\otimes id_G} \\ F' & \xrightarrow[\phi_{G,F'}]{} & F'\otimes G \end{array} \qquad \begin{array}{ccc} K_i(F) & \xrightarrow{K_i(\phi_{G,F})} & K_i(F\otimes G) \\ {\scriptstyle K_i(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle K_i(\phi\otimes id_G)} \\ K_i(F') & \xrightarrow[K_i(\phi_{G,F'})]{} & K_i(F'\otimes G) \end{array}$$

are commutative and so $G\in\Upsilon_1$. ■

COROLLARY 4.2.8 *Let $\{(G_j)_{j\in J}, (\varphi_{j,k})_{j,k\in J}\}$ be an inductive system in $\mathfrak{M}_{\mathbf{C}}$ such that $\varphi_{k,j}$ are injective for all $j,k\in J$, $j<k$, and let $\{G, (\varphi_j)_{j\in J}\}$ be its inductive limit. We*

assume that for all $j,k \in J$, $j < k$,

$$G_j, G_k \in \Upsilon, \qquad \Phi_{i,G_k,F} = K_i\left(id_F \otimes \varphi_{k,j}\right) \circ \Phi_{i,G_j,F}\ .$$

Then

$$G \in \Upsilon, \qquad \Phi_{i,G,F} = K_i\left(id_F \otimes \varphi_j\right) \circ \Phi_{i,G_j,F}$$

for all $j \in J$.

By Corollary 4.2.6, $\{K_i(F \otimes G), (K_i(id_F \otimes \varphi_j))_{j\in J}\}$ is the inductive limit of the inductive system $\{(K_i(F \otimes G_j))_{j\in J}, (K_i\left(id_F \otimes \varphi_{j,k}\right))_{j,k\in J}\}$. By the hypothesis of the Corollary,

$$K_i\left(id_F \otimes \varphi_{k,j}\right) : K_i(F \otimes G_j) \longrightarrow K_i(F \otimes G_k)$$

is a group isomorphism for all $j,k \in J$, $j < k$, so

$$K_i(id_F \otimes \varphi_j) : K_i(F \otimes G_j) \longrightarrow K_i(F \otimes G)$$

is also a group isomorphism for all $j \in J$. Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. The assertion follows from the commutativity of the diagram

$$\begin{tikzcd}
K_i(F)^{p(G_j)} \times K_{i+1}(F)^{q(G_j)} \arrow[r, "K_i(\phi)^{p(G_j)} \times K_{i+1}(\phi)^{q(G_j)}"] \arrow[d, "\Phi_{i,G_j,F}"'] & A \arrow[d, "\Phi_{i,G_j,F'}"'] \\
K_i(F \otimes G_j) \arrow[r, "K_i\left(\phi \otimes G_j\right)"] \arrow[d, "K_i\left(id_F \otimes \varphi_j\right)"'] & K_i(F' \otimes G_j) \arrow[d, "K_i\left(id_{F'} \otimes \varphi_j\right)"'] \\
K_i(F \otimes G) \arrow[r, "K_i(\phi \otimes id_G)"'] & K_i(F' \otimes G)
\end{tikzcd}$$

where $A := K_i(F')^{p(G_j)} \times K_{i+1}(F')^{q(G_j)}$. ∎

DEFINITION 4.2.9 *We denote for every family $(\mathscr{G}_j)_{j\in J}$ of additive groups by $\sum_{j\in J} \mathscr{G}_j$ its direct sum i.e.*

$$\sum_{j\in J} \mathscr{G}_j := \left\{ a \in \prod_{j\in J} \mathscr{G}_j \;\middle|\; \{\, j \in J \mid a_j \neq 0 \,\} \text{ is finite} \right\}.$$

PROPOSITION 4.2.10 *If $(F_j)_{j\in J}$ is a family of E-C*-algebras and F is its C*-direct sum* ([1] Example 4.1.1.6) *then*

$$K_i(F) \approx \sum_{j\in J} K_i(F_j) \ .$$

In particular, the C-direct sum of a family of K-null E-C*-algebras is K-null.*

If J is finite then the assertion follows from Proposition 1.3.3. The general case follows now from the continuity (Axiom 4.2.1). ■

COROLLARY 4.2.11 *If $(\Omega_j)_{j\in J}$ is a family of locally compact spaces and Ω is its topological sum then*

$$K_i(\mathscr{C}_0(\Omega,F)) \approx \sum_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F)) \approx$$

$$\approx \left\{ a \in \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F)) \;\middle|\; \{\, j\in J \mid a_j \neq 0\} \text{ is finite} \right\} .$$

By Proposition 4.2.5, $\mathscr{C}_0(\Omega,F)$ is the direct sum of the family $(\mathscr{C}_0(\Omega_j,F))_{j\in J}$ and the assertion follows from Proposition 4.2.10. ■

PROPOSITION 4.2.12 *If ξ is an ordinal number endowed with its usual topology then* $K_i(\mathscr{C}_0(\xi,F)) \approx \sum\limits_{\eta<\xi} K_i(F)$.

We prove the assertion by transfinite induction. If ξ is not a limit ordinal number then the assertion follows from Corollary 2.3.4 a). Assume ξ is a limit ordinal number and for all $\eta<\zeta<\xi$ let $\varphi_{\zeta,\eta}:\mathscr{C}_0(\eta,F)\longrightarrow\mathscr{C}_0(\zeta,F)$ be the inclusion map. By Proposition 4.2.5, $\mathscr{C}_0(\xi,F)$ may be identified with the inductive limit in $\mathfrak{M}_E$ of the inductive system $\{(\mathscr{C}_0(\eta,\mathrm{F}))_{\eta<\xi},\ (\varphi_{\zeta,\eta})_{\eta<\zeta<\xi}\}$ in $\mathfrak{M}_E$. Thus the assertion follows from the continuity axiom (Axiom 4.2.1) and the induction hypothesis. ■

DEFINITION 4.2.13 *We denote for every $n\in\mathbb{N}$ by $M(n)$ the C*-algebra of $n\times n$-matrices with entries in $\mathbb{C}$.*

AXIOM 4.2.14 (Stability axiom) *There is an $h \in \mathbb{N}$, $h \neq 1$, such that*

$$M(h) \in \Upsilon\,, \qquad p(M(h)) = 1\,, \qquad q(M(h)) = 0\,,$$

$$\Phi_{i,M(h),F} = K_i\,(id_F \otimes \varphi) \circ \Phi_{i,\mathbb{C},F}\,,$$

where

$$\varphi : \mathbb{C} \longrightarrow M(h)\,, \quad \alpha \longmapsto \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

PROPOSITION 4.2.15 *We put for all $j,k \in \mathbb{N}^*$, $j < k$,*

$$\varphi_{k,j} : M(h^j) \longrightarrow M(h^k)\,, \quad x \longmapsto \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

a) For all $j \in \mathbb{N}$,

$$M(h^j) \in \Upsilon\,, \qquad p(M(h^j)) = 1\,, \qquad q(M(h^j)) = 0\,,$$

$$\Phi_{i,M(h^j),F} = K_i\left(id_F \otimes \varphi_{j,0}\right) \circ \Phi_{i,\mathbb{C},F}\,.$$

b) For all $j,k \in \mathbb{N}^*$, $j < k$,

$$\Phi_{i,M(h^k),F} = K_i\left(id_F \otimes \varphi_{k,j}\right) \circ \Phi_{i,M(h^j),F}$$

and $K_i\left(id_F \otimes \varphi_{k,j}\right)$ is a group isomorphism.

a) We prove the assertion by induction with respect to $j \in \mathbb{N}$. For $j = 1$ the assertion is exactly the Stability axiom (Axiom 4.2.14). Let $j > 1$ and assume the assertion holds for $j - 1$. With the notation of Proposition 1.5.4 b),

$$\left(id_{F\otimes M(h)} \otimes \varphi_{(j-1),0}\right) \circ \phi_{\mathbb{C},F\otimes M(h)} \circ \left(id_F \otimes \varphi_{1,0}\right) = id_F \otimes \varphi_{j,0}\,,$$

so by the above and by the induction hypothesis,

$$K_i\left(id_F \otimes \varphi_{j,0}\right) \circ \Phi_{i,\mathbb{C},F} =$$

$$= K_i\left(id_{F\otimes M(h)}\otimes\varphi_{(j-1),0}\right)\circ\Phi_{i,\mathbb{C},F\otimes M(h)}\circ K_i\left(id_F\otimes\varphi_{1,0}\right)\circ\Phi_{i,\mathbb{C},F} =$$

$$= \Phi_{i,M(h^{j-1}),F\otimes M(h)}\circ\Phi_{i,M(h),F}\ .$$

Thus

$$K_i\left(id_F\otimes\varphi_{j,0}\right)\circ\Phi_{i,\mathbb{C},F} : K_i(F)\longrightarrow K_i\left(F\otimes M(h^j)\right)$$

is a group isomorphism. Let $F\xrightarrow{\phi}F'$ be a morphism in $\mathfrak{M}_E$. Since the diagram

$$\begin{tikzcd}
K_i(F) \arrow[r, "\Phi_{i,\mathbb{C},F}"] \arrow[d, "K_i(\phi)"'] & K_i(F\otimes M(1)) \arrow[r, "K_i\left(id_F\otimes\varphi_{j,0}\right)"] \arrow[d, "K_i\left(\phi\otimes id_{M(1)}\right)"] & K_i\left(F\otimes M(h^j)\right) \arrow[d, "K_i\left(\phi\otimes id_{M(h^j)}\right)"] \\
K_i(F') \arrow[r, "\Phi_{i,\mathbb{C},F'}"'] & K_i(F'\otimes M(1)) \arrow[r, "K_i\left(id_{F'}\otimes\varphi(j,0)\right)"'] & K_i\left(F'\otimes M(h^j)\right)
\end{tikzcd}$$

is commutative, we may take

$$\Phi_{i,M(h^j),F} = K_i\left(id_F\otimes\varphi_{j,0}\right)\circ\Phi_{i,\mathbb{C},F}\ .$$

b) By a),

$$K_i\left(id_F\otimes\varphi_{k,j}\right)\circ\Phi_{i,M(h^j),F} = K_i\left(id_F\otimes\varphi_{k,j}\right)\circ K_i\left(id_F\otimes\varphi_{j,0}\right)\circ\Phi_{i,\mathbb{C},F} =$$

$$= K_i\left(id_F\otimes\varphi_{k,0}\right) = \Phi_{i,M(h^k),F}\ .$$

■

THEOREM 4.2.16 *Let H be an infinite-dimensional Hilbert space and $\mathscr{K}(H)$ the C*-algebra of compact operators on H. Then*

$$\mathscr{K}(H)\in\Upsilon,\qquad p(\mathscr{K}(H))=1,\qquad q(\mathscr{K}(H))=0,$$

$$\Phi_{i,\mathscr{K}(H),F} = K_i\left(id_F\otimes\varphi\right)\circ\Phi_{i,\mathbb{C},F}\,,$$

where $\varphi:\mathbb{C}\longrightarrow\mathscr{K}(H)$ is an inclusion map.

Let Ξ be the set of subspaces of H of dimension h^j for some $j\in\mathbb{N}^*$ ordered by inclusion and for every $K\in\Xi$ let π_K be the orthogonal projection of H on K and $G_K := \pi_K\mathscr{K}(H)\pi_K$. We denote for all $K,L\in\Xi$, $K\subset L$, by

$$\varphi_{L,K}: G_K\longrightarrow G_L\,,\qquad\qquad \varphi_K: G_K\longrightarrow\mathscr{K}(H)$$

the inclusion maps. Then $\{(G_K)_{K\in\Xi}\,,\ (\varphi_{L,K})_{L,K\in\Xi}\}$ is an inductive system in $\mathfrak{M}_{\mathbf{C}}$ and $\{\mathscr{K}(H)\,,(\varphi_K)_{K\in\Xi}\}$ is its inductive limit. By Proposition 4.2.15, for $K,L\in\Xi$, $K\subset L$,

$$G_K, G_L \in \Upsilon\,, \qquad p(G_K) = p(G_L) = 1\,, \qquad q(G_K) = q(G_L) = 0\,,$$

$$\Phi_{i,G_L,F} = K_i\left(id_F \otimes \varphi_{L,K}\right) \circ \Phi_{i,G_K,F}\,,$$

and $K_i\left(id_F \otimes \varphi_{L,K}\right)$ is a group isomorphism. By Corollary 4.2.8, for $K\in\Xi$,

$$\mathscr{K}(H) \in \Upsilon\,, \qquad \Phi_{i,\mathscr{K}(H),F} = K_i\left(id_F \otimes \varphi_K\right) \circ \Phi_{i,G_K,F}\,,$$

so $p(\mathscr{K}(H)) = 1$, $q(\mathscr{K}(H)) = 0$. ■

Part II

Projective K-theory

Throughout this part we use the following notation: T is a group, 1 is its neutral element, K is the complex Hilbert space $l^2(T)$, $(T_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subgroups of T the union of which is T, $T_0 := \{1\}$, E is a unital commutative C*-algebra, and f is a Schur E-function for T (Definition 5.0.1).

In the usual K-theory the orthogonal projections (used for K_0) and the unitaries (used for K_1) are identified with elements of the square matrices, which is not a very elegant procedure from the mathematical point of view, but is justified as a very efficient pragmatic solution. It seems to us that in the present more complicated construction the danger of confusion produced by these identifications is greater and we decided to separate these three domains. Unfortunately this separation complicates the presentation and the notation. Moreover, we also do identifications! In general the stability does not hold. We present in Theorem 6.3.3 (as an example) some strong conditions under which stability holds for K_0.

For projective representations of groups we use [2] (but the groups will be finite here) and for the K-theory we use [4], the construction of which we follow step by step. In the sequel we give a list of notation used in this Part.

1) We put for every involutive algebra F,

$$Pr\,F := \left\{ P \in F \mid P = P^* = P^2 \right\}$$

and for every $A \subset F$,

$$A^c := \{ x \in F \mid y \in A \Longrightarrow xy = yx \} .$$

2) We denote for every unital involutive algebra F by 1_F its unit and set

$$Un\,F := \{ U \in F \mid UU^* = U^*U = 1_F \} .$$

3) If F is a unital C*-algebra and $U,V \in Un\,F$ then we denote by $U \sim_h V$ the assertion U and V are homotopic in $Un\,F$ and put

$$Un_0\,F := \{ U \in Un\,F \mid U \sim_h 1_F \} .$$

Moreover $GL(F)$ denotes the group of invertible elements of F and $GL_0(F)$ the elements of $GL(F)$ which are homotopic to 1_F in $GL(F)$.

4) If F is a unital C*-algebra and G is a unital C*-subalgebra of F then we denote by $Un_G F$ the set of elements of $Un\, F$ which are homotopic to an element of $Un\, G$ in $Un\, F$ and by $GL_G(F)$ the set of elements of $GL(F)$ which are homotopic to an element of $GL(G)$ in $GL(F)$.

5) If Ω is a topological space, F a C*-algebra, and $A \subset F$ then we put

$$\mathscr{C}(\Omega, A) := \{\, X \in \mathscr{C}(\Omega, F) \mid \omega \in \Omega \Longrightarrow X(\omega) \in A \,\} \ .$$

6) Hilbert E-C*-algebra ([1] Definition 5.6.1.4).

7) $\mathscr{L}_E(H)$ ([1] Definition 5.6.1.7).

Chapter 5

Some Notation and the Axiom

DEFINITION 5.0.1 Let S be a group and let 1 be its neutral element. A **Schur E-function for S** is a map

$$f : S \times S \longrightarrow Un\,E$$

such that $f(1,1) = 1_E$ and

$$f(r,s)f(rs,t) = f(r,st)f(s,t)$$

for all $r,s,t \in T$. We denote by $\mathscr{F}(S,E)$ the set of Schur E-functions for S.

Schur functions are also called normalized factor set or multiplier or two-co-cycle (for S with values in $Un\,E$) in the literature.

DEFINITION 5.0.2 Let F be an full E-C*-algebra and $n \in \mathbb{N}^*$. We put for every $t \in T_n$, $\xi \in F^{T_n} = F \otimes l^2(T_n)$, and $x \in F$,

$$V_t\xi := V_t^F \xi : T_n \longrightarrow F\,, \quad s \longmapsto f(t,t^{-1}s)\xi(t^{-1}s)\,,$$

$$x \otimes id_K : F^{T_n} \longrightarrow F^{T_n}\,, \quad \xi \longmapsto (x\xi_s)_{s\in T_n}\,,$$

so we have

$$(x \otimes id_K)V_t\xi : T_n \longrightarrow F\,, \quad s \longmapsto f(t,t^{-1}s)x\xi(t^{-1}s)\,.$$

We define

$$F_n := \left\{ \sum_{t\in T_n} (X_t \otimes id_K)V_t \;\middle|\; (X_t)_{t\in T_n} \in F^{T_n} \right\}.$$

If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{C}_E$ then we put

$$\varphi_n : F_n \longrightarrow G_n\,, \quad X \longmapsto \sum_{t\in T_n} ((\varphi X_t) \otimes id_{K_n})V_t\,.$$

F_n is a full E-C*-subalgebra of $\mathscr{L}_F(F^{T_n})$ (Proposition 4.1.7 b), [2] Theorem 2.1.9 h), k)), so $1_{F_n} = 1_E$, and φ_n is an E-C*-homomorphism, injective or surjective if φ is so ([2] Corollary 2.2.5). Moreover F_m is canonically a full E-C*-subalgebra of F_n for every $m \in \mathbb{N}^*$, $m < n$ ([2] Proposition 2.1.2). For every $n \in \mathbb{N}$, $F_n \times G_n \approx (F \times G)_n$.

DEFINITION 5.0.3 We fix in Part II a sequence $(C_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_n$, put

$$A_n := C_n^* C_n\,, \qquad B_n := C_n C_n^*\,,$$

and assume $A_n, B_n \in Pr\,E_n$, $A_n + B_n = 1_E = 1_{E_n}$, and $C_n \in (E_{n-1})^c$ for every $n \in \mathbb{N}$ (where we used the inclusion $E_{n-1} \subset E_n$ in the last relation).

From

$$A_n = A_n(A_n + B_n) = A_n^2 + A_n B_n = A_n + A_n B_n\,,$$

$$C_n = C_n(A_n + B_n) = C_n A_n + C_n B_n = C_n + C_n^2 C_n^*$$

we get $A_n B_n = C_n^2 = 0$ for every $n \in \mathbb{N}$.

We have $C_n \in (F_{n-1})^c$ for every $n \in \mathbb{N}$ and for every full E-C*-algebra F (where we used the inclusion $F_{n-1} \subset F_n$).

DEFINITION 5.0.4 Let $(S_m)_{m\in\mathbb{N}}$ be a sequence of finite groups and $(k_n)_{n\in\mathbb{N}}$ a strictly increasing sequence in $\mathbb{N}$ such that $T_n = \prod_{m=1}^{k_n} S_m$ for all $n \in \mathbb{N}$. We identify S_m with a subgroup of T for every $m \in \mathbb{N}$. Assume that for every $m \in \mathbb{N}$ there is a $g_m \in \mathscr{F}(S_m, E)$ such that

$$f(s,t) = \prod_{m\in\mathbb{N}} g_m(s_m, t_m)$$

for all $s,t \in T$. For every $n \in \mathbb{N}$ let $m \in \mathbb{N}$, $k_{n-1} < m \le k_n$, let $\chi : \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow S_m$ be an injective group homomorphism, and $\beta_1, \beta_2 \in Un\,E$. We put

$$a := \chi(1,0)\,, \qquad b := \chi(0,1)\,, \qquad \alpha_1 := f(a,a)\,, \qquad \alpha_2 := f(b,b)\,,$$

$$C_n := \frac{1}{2}((\beta_1 \otimes id_K)V_a^f + (\beta_2 \otimes id_K)V_b^f)\,.$$

If $f(a,b) = -f(b,a) = 1_E$ and $\alpha_1\beta_1^2 + \alpha_2\beta_2^2 = 0$ then $(C_n)_{n\in\mathbb{N}}$ fulfills the conditions of *Axiom 5.0.3*.

The assertion follows from [2] Theorem 2.2.18 a), b). ■

Remark 1. If $E = \mathbb{C}$, $S_m = \mathbf{Z}_2 \times \mathbf{Z}_2$, and $k_m = m$ for every $m \in \mathbb{N}$ then (by [2] Proposition 3.2.1 c) and [2] Corollary 3.2.2 d)) we may choose $(C_n)_{n\in\mathbb{N}}$ in such a way that the corresponding K-theory coincides with the classical one.

Remark 2. Denote by T_n the set of permutations p of $\mathbb{N}$ such that $\{\, j \in \mathbb{N} \mid p(j) \neq j\} \subset \mathbb{N}_{4n}$ so T is the set of permutations p of $\mathbb{N}$ such that $\{\, j \in \mathbb{N} \mid p(j) \neq j\}$ is finite. This example shows that the given conditions for T_n in Example 5.0.4 are not automatically fulfilled.

Chapter 6

The Functor K_0

6.1 K_0 for $\mathfrak{C}_E$

Throughout this section F denotes a full E-C*-algebra.

PROPOSITION 6.1.1 *Let $n \in \mathbb{N}$.*

a) $A_n, B_n \in (F_{n-1})^c$ *(where we used the inclusion $F_{n-1} \subset F_n$).*

b) $A_n F_n A_n$ *is a unital C*-algebra with A_n as unit.*

c) The map

$$\bar{\rho}_n^F : F_{n-1} \longrightarrow F_n\,, \quad X \longmapsto A_n X = X A_n = A_n X A_n = C_n^* X C_n$$

(where we used the inclusion $F_{n-1} \subset F_n$) is an E-linear injective C-homomorphism.*

Only the injectivity of $\bar{\rho}_n^F$ needs a proof. Let $X \in F_{n-1}$ with $\bar{\rho}_n^F X = 0$. Then

$$C_n^* C_n X = 0\,, \qquad X C_n = C_n X = 0\,,$$

$$X B_n = X C_n C_n^* = 0\,, \qquad X = X(A_n + B_n) = 0\,. \qquad \blacksquare$$

Remark. $\bar{\rho}_n^F$ is not unital since $\bar{\rho}_n^F 1_E = A_n$.

DEFINITION 6.1.2 *We put for all $m, n \in \mathbb{N}$, $m < n$,*

$$\rho_{n,m}^F := \bar{\rho}_n^F \circ \bar{\rho}_{n-1}^F \circ \cdots \circ \bar{\rho}_{m+1}^F : F_m \longrightarrow F_n\,.$$

Then $\{(F_n)_{n\in\mathbb{N}}, (\rho_{n,m}^F)_{n,m\in\mathbb{N}}\}$ is an inductive system of full E-C-algebras with injective E-linear (but not unital) maps. We denote by $\{F_\rightarrow, (\rho_n^F)_{n\in\mathbb{N}}\}$ its algebraic inductive limit. $F_\rightarrow$ is an involutive (but not unital) algebra endowed with the structure of an algebraic E-C*-algebra, ρ_n^F is injective and E-linear for every $n \in \mathbb{N}$, and $(Im\,\rho_n^F)_{n\in\mathbb{N}}$ is an increasing sequence of involutive subalgebras and algebraic E-C*-subalgebras of $F_\rightarrow$ the union of which is $F_\rightarrow$. We put for every $X \in F_n$,*

$$X_\rightarrow := X_{\rightarrow n} := X_{\rightarrow n}^F := \rho_n^F X\,,$$

and

$$1_{\rightarrow n} := 1^F_{\rightarrow n} := \rho^F_n 1_{F_n} = \rho^F_n 1_E\,,$$

$$F_{\rightarrow n} := Im\,\rho^F_n\ .$$

In particular

$$(A_n)_{\rightarrow} = \rho^F_n A_n = 1_{\rightarrow,n-1}, \qquad (B_n)_{\rightarrow} = \rho^F_n B_n, \qquad (C_n)_{\rightarrow} = \rho^F_n C_n\ .$$

We put

$$Pr\,F_{\rightarrow} := \left\{\, P \in F_{\rightarrow} \mid P = P^* = P^2 \,\right\} = \bigcup_{n\in\mathbb{N}} (Pr\,F_{\rightarrow n})\ .$$

For $P,Q \in Pr\,F_{\rightarrow}$ we put $P \sim_0 Q$ if there is an $X \in F_{\rightarrow}$ with $X^*X = P$, $XX^* = Q$ (in this case there is an $n \in \mathbb{N}$ such that $P,Q,X \in F_{\rightarrow n}$); $\sim_0$ is the Murray - von Neumann equivalence relation, which we shall use also in the case of C-algebras. For every $P \in Pr\,F_{\rightarrow}$ we denote by $\dot{P}$ its equivalence class in $Pr\,F/\sim_0$.*

Often we shall identify F_n with $F_{\rightarrow n}$ by using ρ^F_n. By this identification $F_{\rightarrow n}$ is a full E-C*-algebra with $1_{\rightarrow n}$ as unit.

$F_{\rightarrow}$ is also endowed with a C*-norm and its completion in this norm is the C*-inductive limit of the above inductive system, but we shall not use this supplementary structure in the sequel.

PROPOSITION 6.1.3 *If $n \in \mathbb{N}$ and $P \in Pr\,F_{\rightarrow,n-1}$ then*

$$P = (A_n)_{\rightarrow} P \sim_0 (B_n)_{\rightarrow} P = (C_n)_{\rightarrow} P (C_n)^*_{\rightarrow}\ .$$

We have

$$((C_n)_{\rightarrow}P)^*((C_n)_{\rightarrow}P) = P(C_n)^*_{\rightarrow}(C_n)_{\rightarrow}P = (A_n)_{\rightarrow}P\,,$$

$$((C_n)_{\rightarrow}P)((C_n)_{\rightarrow}P)^* = P(C_n)_{\rightarrow}(C_n)^*_{\rightarrow}P = (B_n)_{\rightarrow}P\,,$$

so $(A_n)_{\rightarrow}P \sim_0 (B_n)_{\rightarrow}P$. ■

PROPOSITION 6.1.4 *For every finite family $(P_i)_{i\in I}$ in $Pr\,F_{\rightarrow}$ there is a family $(Q_i)_{i\in I}$ in $Pr\,F_{\rightarrow}$ such that $P_i \sim_0 Q_i$ for every $i \in I$ and $Q_iQ_j = 0$ for all distinct $i,j \in I$.*

We prove the assertion by complete induction with respect to $Card\, I$. Let $i_0 \in I$ and put $J := I \setminus \{i_0\}$. We may assume, by the induction hypothesis, that there is an $n \in \mathbb{N}$ with $P_i \in Pr\, F_{\rightarrow,n-1}$ for all $i \in I$ and $P_i P_j = 0$ for all distinct $i, j \in J$. By Proposition 6.1.3,

$$P_{i_0} = (A_n)_{\rightarrow} P_{i_0} \sim_0 (C_n)_{\rightarrow} P_{i_0} (C_n)^*_{\rightarrow} =: Q_{i_0} ,$$

and

$$Q_{i_0} P_j = (C_n)_{\rightarrow} P_{i_0} (C_n)^*_{\rightarrow} (A_n)_{\rightarrow} P_j = (C_n)_{\rightarrow} P_{i_0} (C_n^* A_n)_{\rightarrow} P_j = 0$$

for all $j \in J$. ■

PROPOSITION 6.1.5 *Let $P, Q \in Pr\, F_{\rightarrow}$.*

a) If $P', P'', Q', Q'' \in Pr\, F_{\rightarrow}$ such that

$$P \sim_0 P' \sim_0 P'', \qquad Q \sim_0 Q' \sim_0 Q'', \qquad P'Q' = P''Q'' = 0$$

then

$$P' + Q' \sim_0 P'' + Q'' .$$

We put

$$\dot{P} \oplus \dot{Q} := \overset{\cdot}{\overbrace{P' + Q'}} .$$

b) $Pr\, F_{\rightarrow} / \sim_0$ endowed with the above composition law $\oplus$ is an additive semi-group with $\dot{0}$ as neutral element. We denote by $K_0(F)$ its associated Grothendieck group and by

$$[\,\cdot\,]_0 : Pr\, F_{\rightarrow} \longrightarrow K_0(F)$$

the Grothendieck map ([4] 3.1.1).

c) $K_0(F) = \{\, [P]_0 - [Q]_0 \mid P, Q \in Pr\, F_{\rightarrow} \}$.

d) For every $a \in K_0(F)$ there are $P, Q \in Pr\, F_{\rightarrow}$ and $n \in \mathbb{N}$ such that

$$P = P(A_n)_{\rightarrow}, \qquad Q = Q(B_n)_{\rightarrow}, \qquad a = [P]_0 - [Q]_0 .$$

a) Let $X, Y \in F_{\rightarrow}$ with

$$X^*X = P', \qquad XX^* = P'', \qquad Y^*Y = Q', \qquad YY^* = Q'' .$$

Then

$$0 = P'Q' = X^*XY^*Y, \qquad 0 = P''Q'' = XX^*YY^*$$

so

$$XY^* = X^*Y = 0, \qquad (X+Y)^*(X+Y) = X^*X + Y^*Y = P' + Q',$$

$$(X+Y)(X+Y)^* = XX^* + YY^* = P'' + Q'', \qquad P' + Q' \sim_0 P'' + Q''.$$

b) and c) follow from a) and Proposition 6.1.4.

d) follows from c) and Proposition 6.1.3. ■

COROLLARY 6.1.6 *The following are equivalent for all $n \in \mathbb{N}$ and $P, Q \in Pr\,F_{\to n}$.*

a) $[P]_0 = [Q]_0$.

b) There is an $R \in Pr\,F_{\to}$ such that

$$PR = QR = 0, \qquad P + R \sim_0 Q + R.$$

c) There is an $m \in \mathbb{N}$, $m > n+1$, such that

$$P + (B_m)_{\to} \sim_0 Q + (B_m)_{\to}$$

or (by identifying F_m with $F_{\to m}$)

$$\left(\prod_{i=n+1}^{m} A_i\right) P + \left(1_E - \prod_{i=n+1}^{m} A_i\right) \sim_0 \left(\prod_{i=n+1}^{m} A_i\right) Q + \left(1_E - \prod_{i=n+1}^{m} A_i\right).$$

$a \Rightarrow b$ follows from Proposition 6.1.4 (and from the definition of the Grothendieck group).

$b \Rightarrow c$. We may assume $R \in F_{\to, m-1}$ for some $m > n+1$. By Proposition 6.1.3,

$$P + (B_m)_{\to}R \sim_0 P + R \sim_0 Q + R \sim_0 Q + (B_m)_{\to}R,$$

so

$$P + (B_m)_{\to} = P + (B_m)_{\to}R + ((B_m)_{\to} - (B_m)_{\to}R) \sim_0$$

$$\sim_0 Q + (B_m)_{\to}R + ((B_m)_{\to} - (B_m)_{\to}R) = Q + (B_m)_{\to}.$$

It follows

$$\left(\prod_{i=n+1}^{m} A_i\right) P + \left(1_E - \prod_{i=n+1}^{m} A_i\right) = \rho^F_{m,n} P + B_m + \left(A_m - \prod_{i=n+1}^{m} A_i\right) \sim_0$$

$$\sim_0 \rho^F_{m,n} Q + B_m + \left(A_m - \prod_{i=n+1}^{m} A_i\right) = \left(\prod_{i=n+1}^{m} A_i\right) Q + \left(1_E - \prod_{i=n+1}^{m} A_i\right) .$$

c⇒a is trivial. ■

COROLLARY 6.1.7 *If for every $n \in \mathbb{N}$ and $P \in Pr\, F_{\to n}$ there is an $m \in \mathbb{N}$, $m > n+1$, such that $P + (B_m)_{\to} \sim_0 1_E$ then $K_0(F) = \{0\}$.*

Let $P, Q \in Pr\, F_{\to}$. By our hypothesis there is an $m \in \mathbb{N}$ such that $P + (B_m)_{\to} \sim_0 Q + (B_m)_{\to}$. By Corollary 6.1.6 $c \Rightarrow a$, $[P]_0 = [Q]_0$. Thus by Proposition 6.1.5 c), $K_0(F) = \{0\}$. ■

COROLLARY 6.1.8 $K_0(E) \neq \{0\}$.

Assume $K_0(E) = \{0\}$. Then $[1_E]_0 = [0]_0$, so by Corollary 6.1.6 $a \Rightarrow c$, there is an $n \in \mathbb{N}$ such that

$$1_E \sim_0 1_E - \prod_{i=1}^{n} A_i .$$

Let ω be a point of the spectrum of E. Since $E_n(\omega)$ is a product of square matrices the above relation leads to a contradiction by using the trace function. ■

PROPOSITION 6.1.9 *Let $\mathscr{G}$ be an additive group and $\nu : Pr\, F_{\to} \to \mathscr{G}$ a map such that*

1) $P, Q \in Pr\, F_{\to}, PQ = 0 \implies \nu(P+Q) = \nu(P) + \nu(Q)$.

2) $P, Q \in Pr\, F_{\to}, P \sim_0 Q \implies \nu(P) = \nu(Q)$.

Then there is a unique group homomorphism $\mu : K_0(F) \to \mathscr{G}$ such that $\mu[P]_0 = \nu(P)$ for every $P \in Pr\, F_{\to}$.

By 2), ν is well-defined on $PrF_{\rightarrow}/\sim_0$ and by 1) and Proposition 6.1.5 a),b), ν is an additive map on $PrF_{\rightarrow}/\sim_0$. By 2) and Corollary 6.1.6 a⇒b, ν is well-defined on $K_0(F)$. The existence and uniqueness of μ with the given properties follows now from Proposition 6.1.5 c). ■

PROPOSITION 6.1.10 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{C}_E$.*

a) For $m,n \in \mathbb{N}$, $m < n$, the diagram

$$\begin{array}{ccc} F_m & \xrightarrow{\rho^F_{n,m}} & F_n \\ {\scriptstyle\varphi_m}\downarrow & & \downarrow{\scriptstyle\varphi_n} \\ G_m & \xrightarrow[\rho^G_{n,m}]{} & G_n \end{array}$$

is commutative. Thus there is a unique E-linear involutive algebra homomorphism $\varphi_{\rightarrow} : F_{\rightarrow} \longrightarrow G_{\rightarrow}$ with

$$\varphi_{\rightarrow} \circ \rho^F_n = \rho^G_n \circ \varphi_n$$

for every $n \in \mathbb{N}$.

b) $\varphi_{\rightarrow}$ is injective or surjective if φ is so.

c) There is a unique group homomorphism $K_0(\varphi) : K_0(F) \longrightarrow K_0(G)$ such that

$$K_0(\varphi)[P]_0 = [\varphi_{\rightarrow} P]_0$$

for every $P \in PrF_{\rightarrow}$.

d) If φ is the identity map then $K_0(\varphi)$ is also the identity map.

e) If $\varphi = 0$ then $K_0(\varphi) = 0$.

a) It is sufficient to prove the assertion for $n = m+1$. For $X \in F_m$,

$$\varphi_n \bar{\rho}^F_n X = \varphi_n(A_n X) = A_n \varphi_n X = \bar{\rho}^G_n \varphi_n X$$

(where we used the inclusion $F_m \subset F_n$).

b) follows from the fact that for every $n \in \mathbb{N}$, φ_n is injective or surjective if φ is so ([2] Theorem 2.1.9 a))).

c) By a) and Proposition 6.1.3, the map

$$Pr\,F_{\rightarrow} \longrightarrow K_0(G), \quad P \longmapsto [\varphi_{\rightarrow} P]_0$$

possesses the properties from Proposition 6.1.9.

d) and e) are obvious. ■

COROLLARY 6.1.11 *If $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ are morphisms in $\mathfrak{C}_E$ then*

$$(\psi \circ \varphi)_{\rightarrow} = \psi_{\rightarrow} \circ \varphi_{\rightarrow}, \qquad K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)\,.$$ ■

PROPOSITION 6.1.12

a) The maps

$$\mu : \check{F} \longrightarrow F\,, \quad (\alpha, x) \longmapsto \alpha + x\,,$$

$$\lambda' : E \longrightarrow \check{F}\,, \quad \alpha \longmapsto (\alpha, -\alpha)$$

are E-C-homomorphisms.*

b)

$$\mu \circ \iota^F = id_F\,, \qquad \iota^F \circ \mu + \lambda' \circ \pi^F = id_{\check{F}}\,,$$

$$K_0(\iota^F) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi^F) = id_{K_0(\check{F})}\,.$$

c)

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\iota^F)} K_0(\check{F}) \underset{K_0(\lambda^F)}{\overset{K_0(\pi^F)}{\rightleftarrows}} K_0(E) \longrightarrow 0$$

is a split exact sequence.

a) is easy to see.

b) For $(\alpha, x), (\beta, y) \in \check{F}$,

$$\iota^F \mu(\alpha, x) = (0, \alpha + x), \qquad \lambda' \pi^F(\alpha, x) = (\alpha, -\alpha)\,,$$

$$(\iota^F \mu(\alpha, x))(\lambda' \pi^F(\beta, y)) = (0, \alpha + x)(\beta, -\beta) = (0, 0)\,,$$

$$(\iota^F \mu + \lambda' \pi^F)(\alpha, x) = (\alpha, x)$$

so $\iota^F \circ \mu + \lambda' \circ \pi^F$ is a full E-C*-homomorphism and

$$\iota^F \circ \mu + \lambda' \circ \pi^F = id_{\check{F}} .$$

By a) and Corollary 6.1.11,

$$\iota^F_{\rightarrow} \circ \mu_{\rightarrow} + \lambda'_{\rightarrow} \circ \pi^F_{\rightarrow} = id_{\check{F}_{\rightarrow}} .$$

By Proposition 6.1.10 c),d) and Corollary 6.1.11, for $P \in Pr\check{F}_{\rightarrow}$,

$$(K_0(\iota^F) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi^F))[P]_0 = K_0(\iota^F \circ \mu)[P]_0 + K_0(\lambda' \circ \pi^F)[P]_0 =$$

$$= [\iota^F_{\rightarrow} \mu_{\rightarrow} P]_0 + [\lambda'_{\rightarrow} \pi^F_{\rightarrow} P]_0 = [(\iota^F \circ \mu + \lambda' \circ \pi^F)_{\rightarrow} P]_0 = [P]_0$$

so by Proposition 6.1.5 c),

$$K_0(\iota^F) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi^F) = id_{K_0(\check{F})} .$$

c) By b), Proposition 6.1.10 d),e), and Corollary 6.1.11,

$$K_0(\pi^F) \circ K_0(\iota^F) = K_0(\pi^F \circ \iota^F) = 0 ,$$

$$K_0(\pi^F) \circ K_0(\lambda^F) = K_0(\pi^F \circ \lambda^F) = id_{K_0(E)} ,$$

$$K_0(\mu) \circ K_0(\iota^F) = K_0(\mu \circ \iota^F) = id_{K_0(F)}$$

and so $K_0(\iota^F)$ is injective. By b), for $a \in K_0(\check{F})$,

$$a = K_0(\iota^F) K_0(\mu) a + K_0(\lambda') K_0(\pi^F) a .$$

Thus if $a \in Ker K_0(\pi^F)$ then $a = K_0(\iota^F) K_0(\mu) a \in Im K_0(\iota^F)$, and so $Ker K_0(\pi^F) = Im K_0(\iota^F)$.

6.2 K_0 for $\mathfrak{M}_E$

DEFINITION 6.2.1 *Let F be an E-C*-algebra and consider the split exact sequence*

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\overleftarrow{\lambda^F}}{\overset{\pi^F}{\rightarrow}} E \longrightarrow 0$$

introduced in Definition 4.1.4. *We put*

$$K_0(F) := Ker K_0(\pi^F) .$$

By Proposition 6.1.12 c), this definition does not contradict the definition given in Proposition 6.1.5 b) for the case that F is an full E-C*-algebra .

$K_0(\{0\}) = \{0\}$ since $\pi^{\{0\}}$ is bijective.

PROPOSITION 6.2.2 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{M}_E$.*

a) The diagram

$$\begin{array}{ccccc} F & \xrightarrow{\iota^F} & \check{F} & \xrightarrow{\pi^F} & E \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\check{\varphi}} & & \| \\ G & \xrightarrow[\iota^G]{} & \check{G} & \xrightarrow[\pi^G]{} & E \end{array}$$

is commutative.

b) The diagram

$$\begin{array}{ccccc} K_0(F) & \xrightarrow{\subset} & K_0(\check{F}) & \xrightarrow{K_0(\pi^F)} & K_0(E) \\ {\scriptstyle K_0(\varphi)}\downarrow & & \downarrow{\scriptstyle K_0(\check{\varphi})} & & \| \\ K_0(G) & \xrightarrow[\subset]{} & K_0(\check{G}) & \xrightarrow[K_0(\pi^G)]{} & K_0(E) \end{array}$$

is commutative, where $K_0(\varphi)$ is defined by $K_0(\check{\varphi})$.

c) If $P \in Pr\, F_{\rightarrow}$ then

$$K_0(\varphi)[P]_0 = [\varphi_{\rightarrow}P]_0 \,.$$

d) $K_0(id_F) = id_{K_0(F)}$.

e) If $\varphi = 0$ then $K_0(\varphi) = 0$.

a) is obvious.

b) By a) and Corollary 6.1.11, the right part of the diagram is commutative. This implies the existence (and uniqueness) of $K_0(\varphi)$.

c) By a), b), Proposition 6.1.10 a),c), and Corollary 6.1.11,

$$K_0(\varphi)[P]_0 = K_0(\check{\varphi})[\iota^F_{\rightarrow}P]_0 = [\check{\varphi}_{\rightarrow}\iota^F_{\rightarrow}P]_0 = [\iota^G_{\rightarrow}\varphi_{\rightarrow}P]_0 = [\varphi_{\rightarrow}P]_0 \,.$$

d) and e) follow from c) and Proposition 6.1.5 c). ■

COROLLARY 6.2.3 *Let $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ be morphisms in $\mathfrak{M}_E$.*

a) $K_0(\psi)\circ K_0(\varphi) = K_0(\psi\circ\varphi)$.

b) If φ is an isomorphism then $K_0(\varphi)$ is also an isomorphism and

$$K_0(\varphi)^{-1} = K_0(\varphi^{-1}) \, .$$

a) follows from Proposition 4.1.5 b), Corollary 6.1.11, and Proposition 6.2.2 b).

b) follows from a) and Proposition 6.2.2 d). ■

PROPOSITION 6.2.4 *For every E-C*-algebra F,*

$$K_0(F) = \left\{ \, [P]_0 - [\sigma^F_{\rightarrow} P]_0 \, \middle| \, P \in Pr\check{F}_{\rightarrow} \right\} \, .$$

For $P \in Pr\check{F}_{\rightarrow}$, by Proposition 6.2.2 c) and Corollary 6.1.11 (since $\pi^F = \pi^F \circ \sigma^F$),

$$K_0(\pi^F)[\sigma^F_{\rightarrow} P]_0 = [\pi^F_{\rightarrow}\sigma^F_{\rightarrow} P]_0 = [\pi^F_{\rightarrow} P]_0 = K_0(\pi^F)[P]_0$$

so

$$[P]_0 - [\sigma^F_{\rightarrow} P]_0 \in Ker\, K_0(\pi^F) = K_0(F) \, .$$

Let $a \in K_0(F)$. By Proposition 6.1.5 d), there are $Q, R \in Pr\check{F}_{\rightarrow}$ and $n \in \mathbb{N}$ such that

$$Q = Q(A_n)_{\rightarrow} \, , \qquad R = R(B_n)_{\rightarrow} \, , \qquad a = [Q]_0 - [R]_0 \, .$$

Then

$$a = [Q(A_n)_{\rightarrow}]_0 + [(B_n)_{\rightarrow} - R(B_n)_{\rightarrow}]_0 - ([R(B_n)_{\rightarrow}]_0 - [(B_n)_{\rightarrow} - R(B_n)_{\rightarrow}]_0) =$$

$$= [Q(A_n)_{\rightarrow} + ((B_n)_{\rightarrow} - R(B_n)_{\rightarrow})]_0 - [(B_n)_{\rightarrow}]_0 \, .$$

If we put

$$P := Q(A_n)_{\rightarrow} + ((B_n)_{\rightarrow} - R(B_n)_{\rightarrow})$$

then

$$a = [P]_0 - [(B_n)_{\rightarrow}]_0 \, .$$

By Proposition 6.2.2 c) and Corollary 6.1.11 (and Definition 4.1.4)

$$0 = K_0(\pi^F)a = K_0(\pi^F)[P]_0 - K_0(\pi^F)[(B_n)_\rightarrow]_0 = [\pi^F_\rightarrow P]_0 - [\pi^F_\rightarrow (B_n)_\rightarrow]_0 ,$$

$$[\sigma^F_\rightarrow P]_0 = [\lambda^F_\rightarrow \pi^F_\rightarrow P]_0 = K_0(\lambda^F)[\pi^F_\rightarrow P]_0 = K_0(\lambda^F)[\pi^F_\rightarrow (B_n)_\rightarrow]_0 =$$

$$= [\lambda^F_\rightarrow \pi^F_\rightarrow (B_n)_\rightarrow]_0 = [\sigma^F_\rightarrow (B_n)_\rightarrow]_0 = [(B_n)_\rightarrow]_0 ,$$

$$a = [P]_0 - [\sigma^F_\rightarrow P]_0 .$$

■

PROPOSITION 6.2.5 *Let F be an full E-C*-algebra and $n \in \mathbb{N}$.*

a) $C_n + C_n^* \in Un_0\, E_n$.

b) For $X, Y \in F_{n-1}$,

$$(C_n + C_n^*)(A_n X + B_n Y)(C_n + C_n^*) = B_n X + A_n Y .$$

c) If $U, V \in Un\, F_{n-1}$ *then* $A_n U + B_n V \in Un\, F_n$.

d) If $U \in Un\, F_{n-1}$ *then* $A_n U + B_n \in Un\, F_n$ *and* $A_n U + B_n U^* \in Un_0\, F_n$.

a) From

$$(C_n + C_n^*)(C_n + C_n^*) = B_n + A_n = 1_E$$

it follows that $C_n + C_n^*$ is unitary. Being selfadjoint, its spectrum is contained in $\{-1, +1\}$ and so it belongs to $Un_0\, E_n$ ([4] Lemma 2.1.3 (ii)).

b) We have

$$(C_n + C_n^*)(A_n X + B_n Y)(C_n + C_n^*) = (C_n X + C_n^* Y)(C_n + C_n^*) = B_n X + A_n Y .$$

c) We have

$$(A_n U + B_n V)(A_n U + B_n V)^* = A_n + B_n = 1_E ,$$

$$(A_n U + B_n V)^*(A_n U + B_n V) = A_n + B_n = 1_E .$$

d) By c), $A_n U + B_n \in Un\, F_n$. By b),

$$(C_n + C_n^*)(A_n U^* + B_n)(C_n + C_n^*) = B_n U^* + A_n ,$$

so it follows from a), that $A_nU^* + B_n$ is homotopic to $B_nU^* + A_n$ in $Un\, F_n$ and so

$$A_nU + B_nU^* = (A_nU + B_n)(A_n + B_nU^*)$$

is homotopic in $Un\, F_n$ to

$$(A_nU + B_n)(A_nU^* + B_n) = A_n + B_n = 1_E\,,$$

i.e. $A_nU + B_nU^* \in Un_0\, F_n$. ■

PROPOSITION 6.2.6 *Let F be a full E-C*-algebra, $n \in \mathbb{N}$, $P, Q \in Pr\, F_n$, and $X \in F_n$ with $X^*X = P$, $XX^* = Q$. Then there is a $U \in Un_0\, F_{n+2}$ with*

$$U(A_{n+2}A_{n+1}P)U^* = A_{n+2}A_{n+1}Q\,, \qquad \text{i.e.} \qquad U_{\rightarrow}P_{\rightarrow}U^*_{\rightarrow} = Q_{\rightarrow}\ .$$

We have $X(1_E - P) = (1_E - Q)X = 0$. Put

$$V := A_{n+1}X + C_{n+1}(1_E - P) + C^*_{n+1}(1_E - Q) + B_{n+1}X^* \qquad (\in F_{n+1})\ .$$

Then

$$V^* = A_{n+1}X^* + C^*_{n+1}(1_E - P) + C_{n+1}(1_E - Q) + B_{n+1}X\,,$$

$$VV^* = A_{n+1}Q + B_{n+1}(1_E - P) + A_{n+1}(1_E - Q) + B_{n+1}P = A_{n+1} + B_{n+1} = 1_E\,,$$

$$V^*V = A_{n+1}P + A_{n+1}(1_E - P) + B_{n+1}(1_E - Q) + B_{n+1}Q = A_{n+1} + B_{n+1} = 1_E$$

so $V \in Un\, F_{n+1}$. Moreover

$$VA_{n+1}P = A_{n+1}X\,, \qquad\qquad A_{n+1}XV^* = A_{n+1}Q\ .$$

Put

$$U := A_{n+2}V + B_{n+2}V^*\ .$$

By Proposition 6.2.5 d), $U \in Un_0\, F_{n+2}$. We have

$$U(A_{n+2}A_{n+1}P)U^* = (A_{n+2}V + B_{n+2}V^*)A_{n+2}A_{n+1}P(A_{n+2}V^* + B_{n+2}V) =$$

$$= A_{n+2}A_{n+1}X(A_{n+2}V^* + B_{n+2}V) = A_{n+2}A_{n+1}Q\ .$$ ■

PROPOSITION 6.2.7 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{M}_E$ and $a \in Ker\, K_0(\varphi)$.*

a) There are $n\in\mathbb{N}$, $P\in Pr\check{F}_{\rightarrow n}$, *and* $U\in Un_0\,\check{G}_{\rightarrow,n+2}$ *such that*

$$a=[P]_0-[\sigma^F_{\rightarrow}P]_0 \qquad\qquad U(\check{\varphi}_{\rightarrow}P)U^*=\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow}P\,.$$

b) If φ *is surjective then there is a* $P\in Pr\check{F}_{\rightarrow}$ *such that*

$$a=[P]_0-[\sigma^F_{\rightarrow}P]_0, \qquad\qquad \check{\varphi}_{\rightarrow}P=\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow}P\,.$$

a) By Proposition 6.2.4, there are $m\in\mathbb{N}$ and $Q\in Pr\check{F}_{\rightarrow,m-1}$ such that

$$a=[Q]_0-[\sigma^F_{\rightarrow}Q]_0\,.$$

Since $\check{\varphi}\circ\sigma^F=\sigma^G\circ\check{\varphi}$, by Proposition 6.1.10 c) and Corollary 6.1.11,

$$0=K_0(\varphi)a=[\check{\varphi}_{\rightarrow}Q]_0-[\check{\varphi}_{\rightarrow}\sigma^F_{\rightarrow}Q]_0=[\check{\varphi}_{\rightarrow}Q]_0-[\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow}Q]_0\,.$$

By Corollary 6.1.6 a⇒c, there is an $n\in\mathbb{N}$, $n>m$, such that

$$\check{\varphi}_{\rightarrow}Q+(B_n)_{\rightarrow}\sim_0\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow}Q+(B_n)_{\rightarrow}=\sigma^G_{\rightarrow}(\check{\varphi}_{\rightarrow}Q+(B_n)_{\rightarrow})\,.$$

Put

$$P:=Q+(B_n)_{\rightarrow}\in Pr\check{F}_{\rightarrow n}\,.$$

Then

$$[P]_0-[\sigma^F_{\rightarrow}P]_0=[Q]_0+[(B_n)_{\rightarrow}]_0-[\sigma^F_{\rightarrow}Q]_0-[(B_n)_{\rightarrow}]_0=a\,,$$
$$[\check{\varphi}_{\rightarrow}P]_0-[\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow}P]_0=[\check{\varphi}_{\rightarrow}Q]_0+[(B_n)_{\rightarrow}]_0-[\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow}Q]_0-[(B_n)_{\rightarrow}]_0=0\,.$$

By Corollary 6.1.6 a⇒b and Proposition 6.2.6, there is a $U\in Un_0\,\check{G}_{\rightarrow,n+2}$ with

$$U(\check{\varphi}_{\rightarrow}P)U^*=\sigma^G_{\rightarrow}\check{\varphi}P\,.$$

b) By a), there are $n\in\mathbb{N}$, $n>2$, $Q\in Pr\check{F}_{\rightarrow,n-2}$, and $U\in Un_0\,\check{G}_{\rightarrow n}$ such that

$$a=[Q]_0-[\sigma^F_{\rightarrow}Q]_0, \qquad\qquad U(\check{\varphi}_{\rightarrow}Q)U^*=\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow}Q\,.$$

Since $\varphi_n:\check{F}_n\longrightarrow\check{G}_n$ is surjective, by [4] Lemma 2.1.7 (i), there is a $V\in Un\,\check{F}_{\rightarrow n}$ with $\check{\varphi}_nV=U$. We put

$$P:=VQV^*\sim_0 Q$$

so

$$a=[P]_0-[\sigma^F_{\rightarrow}P]_0$$

and

$$\check{\varphi}_{\to}P = (\check{\varphi}_{\to}V)(\check{\varphi}_{\to}Q)(\check{\varphi}_{\to}V^*) = U(\check{\varphi}_{\to}Q)U^* = \sigma^G_{\to}\check{\varphi}_{\to}Q\,,$$

$$\sigma^G_{\to}\check{\varphi}_{\to}P = \sigma^G_{\to}\check{\varphi}_{\to}Q = \check{\varphi}_{\to}P\,.$$

■

PROPOSITION 6.2.8 *Let*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$.

a) $\check{\varphi}_{\to}$ *is injective.*

b) The following are equivalent for all $X \in \check{G}_{\to}$*:*

b_1) $X \in Im\,\check{\varphi}_{\to}$.

b_2) $\check{\psi}_{\to}X = \sigma^H_{\to}\check{\psi}_{\to}X$.

c) $K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \xrightarrow{K_0(\psi)} K_0(H)$ *is exact.*

a) $\check{\varphi}$ is injective (Proposition 4.1.5 a)) and the assertion follows from Proposition 6.1.10 b).

$b_1 \Rightarrow b_2$ follows from $\psi \circ \varphi = 0$.

$b_2 \Rightarrow b_1$. Let $n \in \mathbb{N}$ such that $X \in \check{G}_{\to n}$, which we identify with $\check{G}_n$. Then X has the form

$$X = \sum_{t \in T_n} ((\alpha_t, Y_t) \otimes id_K) V_t^{\check{G}}\,,$$

where $(\alpha_t, Y_t) \in \check{G}$ for every $t \in T_n$, and so by b_2),

$$\sum_{t \in T_n} ((\alpha_t, \psi Y_t) \otimes id_K) V_t^{\check{H}} = \check{\psi}_n X = \sigma^H_n \check{\psi}_n X = \sum_{t \in T_n} ((\alpha_t, 0) \otimes id_K) V_t^{\check{H}}\,.$$

It follows $\psi Y_t = 0$ for every $t \in T_n$ ([2] Theorem 2.1.9 a)). Thus for every $t \in T_n$ there is a $Z_t \in F$ with $\varphi Z_t = Y_t$ and we get

$$X = \sum_{t \in T_n} ((\alpha_t, \varphi Z_t) \otimes id_K) V_t^{\check{G}} =$$

$$= \check{\varphi}_n \left(\sum_{t \in T_n} ((\alpha_t, Z_t) \otimes id_K) V_t^{\check{F}} \right) \in Im\, \check{\varphi}_n \subset Im\, \check{\varphi}_{\rightarrow} \,.$$

c) By Corollary 6.2.3 a) and Proposition 6.2.2 e),

$$K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi) = 0$$

so $Im\, K_0(\varphi) \subset Ker\, K_0(\psi)$. Let $a \in Ker\, K_0(\psi)$. By Proposition 6.2.7 b), there is a $P \in Pr\, \check{G}_{\rightarrow}$ such that

$$a = [P]_0 - [\sigma^G_{\rightarrow} P]_0, \qquad \check{\psi}_{\rightarrow} P = \sigma^H_{\rightarrow} \check{\psi}_{\rightarrow} P \,.$$

Then P has the form

$$P = \sum_{t \in T_n} ((\alpha_t, X_t) \otimes id_K) V_t^{\check{G}}$$

for some $n \in \mathbb{N}$ with $(\alpha_t, X_t) \in E \times G$ for every $t \in T_n$, where we identified $\check{G}_n$ with $\check{G}_{\rightarrow n}$. We get

$$\sum_{t \in T_n} ((\alpha_t, \psi X_t) \otimes id_K) V_t^{\check{H}} = \check{\psi}_{\rightarrow} P = \sigma^H_{\rightarrow} \check{\psi}_{\rightarrow} P = \sum_{t \in T_n} ((\alpha_t, 0) \otimes id_K) V_t^{\check{H}} \,.$$

Thus $\psi X_t = 0$ ([2] Theorem 2.1.9 a)) and there is an $Y_t \in F$ with $\varphi Y_t = X_t$ for every $t \in T_n$. We put

$$Q := \sum_{t \in T_n} ((\alpha_t, Y_t) \otimes id_K) V_t^{\check{F}} \in Pr\, \check{F}_{\rightarrow}$$

with the usual identification ($\check{\varphi}$ is an embedding !). Then

$$\check{\varphi}_{\rightarrow} Q = \sum_{t \in T_n} ((\alpha_t, \varphi Y_t) \otimes id_K) V_t^{\check{G}} = \sum_{t \in T_n} ((\alpha_t, X_t) \otimes id_K) V_t^{\check{G}} = P$$

and by Proposition 6.2.2 c) (since $\check{\varphi} \circ \sigma^F = \sigma^G \circ \check{\varphi}$),

$$K_0(\varphi)([Q]_0 - [\sigma^F_{\rightarrow} Q]_0) = [\check{\varphi}_{\rightarrow} Q]_0 - [\check{\varphi}_{\rightarrow} \sigma^F_{\rightarrow} Q]_0 =$$

$$= [\check{\varphi}_{\rightarrow} Q]_0 - [\sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} Q]_0 = [P]_0 - [\sigma^G_{\rightarrow} P]_0 = a \,.$$

Thus $Ker\, K_0(\psi) \subset Im\, K_0(\varphi)$, $Ker\, K_0(\psi) = Im\, K_0(\varphi)$. ■

PROPOSITION 6.2.9 (Split Exact Theorem for K_0) *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\underset{\lambda}{\longleftarrow}}{\overset{\psi}{\longrightarrow}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ *then*

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \underset{\xleftarrow[K_0(\lambda)]{}}{\xrightarrow{K_0(\psi)}} K_0(H) \longrightarrow 0$$

is also split exact. In particular the map

$$K_0(F) \times K_0(H) \longrightarrow K_0(G), \quad (a,b) \longmapsto K_0(\varphi)a + K_0(\lambda)b$$

is a group isomorphism and $K_0(\check{F}) \approx K_0(E) \times K_0(F)$ *for every* E*-C*-algebra* F.

By Proposition 6.2.8 c), the second sequence is exact at $K_0(G)$. From

$$K_0(\psi) \circ K_0(\lambda) = K_0(\psi \circ \lambda) = K_0(id_H) = id_{K_0(H)}$$

(Corollary 6.2.3 a) and Proposition 6.2.2 d)) it follows that this sequence is (split) exact at $K_0(H)$.

Let $a \in Ker K_0(\varphi)$. By Proposition 6.2.7 a), there are $n \in \mathbb{N}$, $P \in Pr\check{F}_{\rightarrow n}$, and $U \in Un_0\, \check{G}_{\rightarrow,n+2}$ such that

$$a = [P]_0 - [\sigma^F_{\rightarrow} P]_0, \qquad U(\check{\varphi}_{\rightarrow} P)U^* = \sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P\ .$$

Put

$$V := (\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U^*)U \in Un\, \check{G}_{\rightarrow,n+2}\ .$$

Then

$$\check{\psi}_{\rightarrow} V = (\check{\psi}_{\rightarrow} U^*)(\check{\psi}_{\rightarrow} U) = 1_{\rightarrow,n+2}, \qquad \sigma^H_{\rightarrow} \check{\psi}_{\rightarrow} V = \check{\psi}_{\rightarrow} V\ .$$

By Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $W \in Un\, \check{F}_{\rightarrow,n+2}$ with $\check{\varphi}_{\rightarrow} W = V$ ($\check{\varphi}$ is an embedding). We have

$$\check{\varphi}_{\rightarrow}(WPW^*) = V(\check{\varphi}_{\rightarrow} P)V^* = (\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U^*)U(\check{\varphi}_{\rightarrow} P)U^*(\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U) =$$

$$= (\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U^*)(\sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P)(\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U) = \check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow}(U^*(\sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P)U) =$$

$$= \check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} \check{\varphi}_{\rightarrow} P = \sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P = \check{\varphi}_{\rightarrow} \sigma^F_{\rightarrow} P\ .$$

Since $\check{\varphi}_{\rightarrow}$ is injective (Proposition 6.2.8 a)),

$$P \sim_0 WPW^* = \sigma^F_{\rightarrow} P, \qquad a = 0$$

and $K_0(\varphi)$ is injective.

The last assertion follows since

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\xleftarrow{\lambda^F}}{\xrightarrow{\pi^F}} E \longrightarrow 0$$

is a split exact sequence. ■

COROLLARY 6.2.10 *Let F,G be E-C*-algebras.*

a) If we put

$$\iota_1 : F \longrightarrow F \times G, \quad x \longmapsto (x,0), \qquad \pi_1 : F \times G \longrightarrow F, \quad (x,y) \longmapsto x,$$

$$\iota_2 : G \longrightarrow F \times G, \quad y \longmapsto (0,y), \qquad \pi_2 : F \times G \longrightarrow F, \quad (x,y) \longmapsto y,$$

then the sequences

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\iota_1)} K_0(F \times G) \underset{\xleftarrow{K_0(\iota_2)}}{\xrightarrow{K_0(\pi_2)}} K_0(G) \longrightarrow 0,$$

$$0 \longrightarrow K_0(G) \xrightarrow{K_0(\iota_2)} K_0(F \times G) \underset{\xleftarrow{K_0(\iota_1)}}{\xrightarrow{K_0(\pi_1)}} K_0(F) \longrightarrow 0$$

are split exact.

b) The map

$$K_0(F) \times K_0(G) \longrightarrow K_0(F \times G), \quad (a,b) \longmapsto K_0(\iota_1)a + K_0(\iota_2)b$$

is a group isomorphism **(Product Theorem for K_0)**.

a) is easy to see.

b) follows from a) and Proposition 6.2.9. ■

THEOREM 6.2.11 (Homotopy invariance of K_0)

a) If $\varphi, \psi : F \longrightarrow G$ are homotopic morphisms in $\mathfrak{M}_E$, then $K_0(\varphi) = K_0(\psi)$.

b) If $F \xrightarrow{\varphi} G$, $G \xrightarrow{\psi} F$ is a homotopy in $\mathfrak{M}_E$ then

$$K_0(\varphi) \circ K_0(\psi) = id_{K_0(G)}, \qquad K_0(\psi) \circ K_0(\varphi) = id_{K_0(F)}.$$

c) If F and G are homotopic E-C-algebras then $K_0(F)$ and $K_0(G)$ are isomorphic.*

d) If F is an E-C-algebra such that id_F is homotopic to*

$$0_F : F \longrightarrow F\,, \quad x \longmapsto 0$$

then F is homotopic to $\{0\}$.

e) If the E-C-algebra F is homotopic to $\{0\}$ then $K_0(F) = \{0\}$.*

a) Let

$$\phi_s : F \longrightarrow G, \qquad\qquad s \in [0,1]$$

be a pointwise continuous path of morphisms in $\mathfrak{M}_E$ such that $\phi_0 = \varphi$, $\phi_1 = \psi$. Then

$$\check{\phi}_s : \check{F} \longrightarrow \check{G}, \qquad\qquad s \in [0,1]$$

is a pointwise continuous path of morphisms in $\mathfrak{C}_E$ with $\check{\phi}_0 = \check{\varphi}$, $\check{\phi}_1 = \check{\psi}$ and for every $n \in \mathbb{N}$,

$$(\check{\phi}_s)_{\to n} : (\check{F})_{\to n} \longrightarrow (\check{G})_{\to n}, \qquad\qquad s \in [0,1]$$

is a pointwise continuous path in $\mathfrak{C}_E$ with $(\check{\phi}_0)_{\to n} = (\check{\varphi})_{\to n}$ and $(\check{\phi}_1)_{\to n} = (\check{\psi})_{\to n}$. For every $P \in Pr\,\check{F}_{\to n}$,

$$[0,1] \longrightarrow Pr\,(\check{G})_{\to n}, \quad s \longmapsto (\check{\phi}_s)_{\to n}P$$

is continuous so (by [4] Proposition 2.2.7)

$$K_0(\varphi)[P]_0 = [\varphi_{\to}P]_0 = [\psi_{\to}P]_0 = K_0(\psi)[P]_0$$

(Proposition 6.2.2 c)). By Proposition 6.2.4, $K_0(\varphi) = K_0(\psi)$.

b) follows from a), Corollary 6.2.3 a), and Proposition 6.2.2 d).

c) follows from b).

d) If we put $\varphi : F \longrightarrow \{0\}$ and $\psi : \{0\} \longrightarrow F$ then $\psi \circ \varphi = 0_F$ is homotopic to id_F and $\varphi \circ \psi$ is homotopic to $id_{\{0\}}$, so F is homotopic to $\{0\}$.

e) follows from c). ∎

We show now that K_0 is continuous with respect to inductive limits.

THEOREM 6.2.12 (Continuity of K_0) *Let $\{(F_i)_{i\in I}, (\varphi_{ij})_{i,j\in I}\}$ be an inductive system in $\mathfrak{M}_E$ and let $\{F, (\varphi_i)_{i\in I}\}$ be its inductive limit in $\mathfrak{M}_E$. By* Corollary 6.2.3 a)*,*

$$\{(K_0(F_i))_{i\in I}, (K_0(\varphi_{ij}))_{i,j\in I}\}$$

is an inductive system in the category of additive groups. Let $\{\mathscr{G}, (\psi_i)_{i\in I}\}$ be its limit in this category and let $\psi : \mathscr{G} \longrightarrow K_0(F)$ be the group homomorphism such that $\psi \circ \psi_i = K_0(\varphi_i)$ for every $i \in I$. Then ψ is a group isomorphism.

$\{(\check{F}_i)_{i\in I}, (\check{\varphi}_{ij})_{i,j\in I}\}$ is an inductive system in $\mathfrak{C}_E$ and by [2] Proposition 1.2.9 b), $\{\check{F}, (\check{\varphi}_i)_{i\in I}\}$ may be identified with its inductive limit in $\mathfrak{C}_E$. By[2] Proposition 2.3.5, for every $n \in \mathbb{N}$, $\{((\check{F}_i)_{\rightarrow n})_{i\in I}, ((\check{\varphi}_{ij})_{\rightarrow n})_{i,j\in I}\}$ is an inductive system in $\mathfrak{C}_E$ and $\{(\check{F}_{\rightarrow n}, ((\check{\varphi}_i)_{\rightarrow n})_{i\in I}\}$ may be identified with its inductive limit in $\mathfrak{C}_E$.

Step 1 ψ is surjective

Let $Q \in Pr(\check{F})_{\rightarrow n}$. By [5] L.2.2, there are $i \in I$ and $P \in Pr(\check{F}_i)_{\rightarrow n}$ such that $\|(\check{\varphi}_i)_{\rightarrow n}P - Q\| < 1$, so by [4] Proposition 2.2.4, $(\check{\varphi}_i)_{\rightarrow n}P \sim_0 Q$. By Proposition 6.2.2 b),c)

$$\psi\psi_i[P]_0 = K_0(\varphi_i)[P]_0 = K_0(\check{\varphi}_i)[P]_0 = [(\check{\varphi}_i)_{\rightarrow n}P]_0 = [Q]_0\,.$$

Since

$$Pr\check{F}_{\rightarrow} = \bigcup_{n\in\mathbb{N}} Pr(\check{F})_{\rightarrow n}\,,$$

ψ is surjective.

Step 2 ψ is injective

Let $a \in \mathscr{G}$ with $\psi a = 0$. Since $\mathscr{G} = \bigcup_{i\in I} Im\,\psi_i$, there is an $i \in I$ and an $a_i \in K_0(F_i)$ with $a = \psi_i a_i$. There are $n \in \mathbb{N}$ and $P, Q \in Pr(\check{F}_i)_{\rightarrow n}$ such that

$$a_i = [P]_0 - [Q]_0$$

(by Proposition 6.1.5 c)). By Proposition 6.2.2 c),

$$0 = \psi a = \psi\psi_i a = K_0(\varphi_i)a = K_0(\varphi_i)[P]_0 - K_0(\varphi_i)[Q]_0 =$$

$$= [(\check{\varphi}_i)_{\rightarrow n}P]_0 - [(\check{\varphi}_i)_{\rightarrow n}Q]_0 \,.$$

By Corollary 6.1.6 a⇒b, there is an $R \in Pr\,(\check{F}_i)_{\rightarrow}$ such that

$$PR = QR = 0, \qquad\qquad P+R \sim_0 Q+R$$

and we get

$$a = [P]_0 + [R]_0 - [Q]_0 - [R]_0 = [P+R]_0 - [Q+R]_0 = 0\,.$$

■

6.3 Stability of K_0

The stability of K_0 holds only under strong supplementary hypotheses. We present below such possible hypotheses, which we fix for this section. We shell give only a sketch of the proof.

Let S be a finite group, $\chi : \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow S$ an injective group homomorphism,

$$a := \omega(1,0), \qquad b := \omega(0,1), \qquad c := \omega(1,1)\,,$$

and g a Schur E-function for S such that

$$g(a,b) = g(a,c) = g(b,c) = -g(b,a) = 1_E\,.$$

We put for every $n \in \mathbb{N}$,

$$T_n := S^n = \left\{\, t \in S^{\mathbb{N}} \;\middle|\; m \in \mathbb{N}, m > n \Rightarrow t_m = 1 \,\right\},$$

$$T := \bigcup_{n\in\mathbb{N}} T_n = \left\{\, t \in S^{\mathbb{N}} \;\middle|\; \{n \in \mathbb{N}, t_n \neq 1\} \text{ is finite} \,\right\},$$

$$f : T \times T \longrightarrow E\,, \quad (s,t) \longmapsto \prod_{n\in\mathbb{N}} g(s_n,t_n)\,,$$

$$\overset{n}{s} : \mathbb{N} \longrightarrow S\,, \quad m \longmapsto \begin{cases} s & \text{if} \quad m = n \\ 1 & \text{if} \quad m \neq n \end{cases},$$

for every $s \in S$, and

$$C_n := \frac{1}{2}(V^f_{\overset{n}{a}} + V^f_{\overset{n}{b}}), \qquad A_n := C_n^* C_n, \qquad B_n := C_n C_n^*\,.$$

Then f is a Schur E-function for T and the following hold for all $s,t\in S$ and $n\in\mathbb{N}$:

$$f(\overset{n}{s},\overset{n}{t}) = g(s,t)\,,$$

$$\overset{n}{t}=1 \implies V^f_{\overset{n}{s}}V^f_{\overset{n}{t}} = V^f_{\overset{n}{t}}V^f_{\overset{n}{s}}\,,$$

$$s\in T_{n-1} \implies V^f_s V_{\overset{n}{t}} = V_{\overset{n}{t}}V^f_s\,,$$

$$A_n = \frac{1}{2}(V^f_1 + V^f_{\overset{n}{c}}) \in PrE_n, \qquad B_n = \frac{1}{2}(V^f_1 - V^f_{\overset{n}{c}}) \in PrE_n\,,$$

$$A_n + B_n = V^f_1 = 1_E\,,$$

so the assumptions of Axiom 5.0.3 are fulfilled.

Remark. If χ is bijective and $E=\mathbb{C}$ then the corresponding projective K-theory coincides with the usual K-theory.

PROPOSITION 6.3.1 *Let F be an full E-C*-algebra and $m,n\in\mathbb{N}$. We define*

$$\alpha := \alpha^F_{m,n} : (F_m)_n \longrightarrow F_{m+n}\,,$$

$$\beta := \beta^F_{m,n} : F_{m+n} \longrightarrow (F_m)_n\,,$$

by

$$(\alpha X)_{(s,t)} := (X_t)_s, \qquad ((\beta Y)_t)_s := Y_{(s,t)}$$

for every $X\in(F_m)_n$, $Y\in F_{m+n}$, and $(s,t)\in S^m\times S^n = S^{m+n}$, where the identification is given by the bijective map

$$S^m\times S^n \longrightarrow S^{m+n}, \quad (s,t)\longmapsto (s_1,\cdots,s_m,t_1,\cdots,t_n)\,.$$

a) α and β are E-C-isomorphisms and $\alpha=\beta^{-1}$.*

b) $\alpha A_n = A_{m+n}$.

c) The diagram

$$\begin{array}{ccc}
(F_m)_{n-1} & \xrightarrow{\alpha^F_{m,n-1}} & F_{m+n-1} \\
{\scriptstyle \bar\rho^{F_m}_n}\big\downarrow & & \big\downarrow{\scriptstyle \bar\rho^F_{m+n}} \\
(F_m)_n & \xrightarrow[\alpha^F_{m,n}]{} & F_{m+n}
\end{array}$$

is commutative.

It is obvious that α and β are E-linear and $\alpha \circ \beta = id_{F_{m+n}}$, $\beta \circ \alpha = id_{(F_m)_n}$. Thus α and β are bijective and $\alpha = \beta^{-1}$.

For $X, Y \in (F_m)_n$ and $(s,t) \in S^m \times S^n$, by [2] Theorem 2.1.9 c),g),

$$(\alpha X^*)_{(s,t)} = ((X^*)_t)_s = (\tilde{f}(t)(X_{t^{-1}})^*)_s = \tilde{f}(s)\tilde{f}(t)((X_{t^{-1}})_{s^{-1}})^* =$$

$$= \tilde{f}((s,t))(\alpha X_{(s,t)^{-1}})^* = ((\alpha X)^*)_{(s,t)},$$

$$((\alpha X)(\alpha Y))_{(s,t)} =$$

$$= \sum_{(u,v) \in S^m \times S^n} f((u,v),(u^{-1}s, v^{-1}t))(\alpha X)_{(u,v)}(\alpha Y)_{(u^{-1}s, v^{-1}t)} =$$

$$= \sum_{(u,v) \in S^m \times S^n} f(u, u^{-1}s) f(v, v^{-1}t)(X_v)_u (Y_{v^{-1}t})_{u^{-1}s} =$$

$$= \sum_{v \in S^n} f(v, v^{-1}t)(X_v Y_{v^{-1}t})_s =$$

$$= \left(\sum_{v \in S^n} f(v, v^{-1}t) X_v Y_{v^{-1}t} \right)_s = ((XY)_t)_s = (\alpha(XY))_{(s,t)}$$

so α is a C*-homomorphism and the assertion follows.

b) follows from the definition of A_n and A_{m+n}.

c) follows from b). ■

PROPOSITION 6.3.2 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{C}_E$ and $m, n \in \mathbb{N}$. With the notation of* Proposition 6.3.1 *the diagram*

$$\begin{array}{ccc} (F_m)_n & \xrightarrow{\alpha^F_{m,n}} & F_{m+n} \\ {\scriptstyle (\varphi_m)_n}\downarrow & & \downarrow{\scriptstyle \varphi_{m+n}} \\ (G_m)_n & \xrightarrow[\alpha^G_{m,n}]{} & G_{m+n} \end{array}$$

is commutative.

For $X \in (F_m)_n$ and $(s,t) \in S^m \times S^n = S^{m+n}$,

$$(\varphi_{m+n}\alpha^F_{m,n}X)_{(s,t)} = \varphi(\alpha^F_{m,n}X)_{(s,t)} = \varphi(X_t)_s =$$

$$= (\varphi_m X_t)_s = (((\varphi_m)_n X)_t)_s = (\alpha^G_{m,n}(\varphi_m)_n X)_{(s,t)}$$

so

$$\varphi_{m+n} \circ \alpha^F_{m,n} = \alpha^G_{m,n} \circ (\varphi_m)_n \; . \qquad \blacksquare$$

THEOREM 6.3.3 (Stability for K_0) *If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ and $n \in \mathbb{N}$ then*

$$K_0(F_n) \approx K_0(F), \qquad K_0(G_n) \approx K_0(G), \qquad K_0(\varphi_n) \approx K_0(\varphi) \; . \qquad \blacksquare$$

Remark. If $(F_\infty, (\rho^F_n)_{n\in\mathbb{N}})$ and $(G_\infty, (\rho^G_n)_{n\in\mathbb{N}})$ denote the inductive limits in $\mathfrak{M}_E$ of the corresponding inductive systems $((F_n)_{n\in\mathbb{N}}, (\rho^F_{n,m})_{n,m\in\mathbb{N}})$ and $((G_n)_{n\in\mathbb{N}}, (\rho^G_{n,m})_{n,m\in\mathbb{N}})$ then, with obvious notation,

$$K_0(F_\infty) \approx K_0(F), \qquad K_0(G_\infty) \approx K_0(G), \qquad K_0(\varphi_\infty) \approx K_0(\varphi) \; .$$

Chapter 7

The Functor K_1

7.1 Definition of K_1

PROPOSITION 7.1.1 *If F is a full E-C^*-algebra and $n \in \mathbb{N}$ then*

$$\bar{\tau}_n^F : Un\,F_{n-1} \longrightarrow Un\,F_n\,, \quad U \longmapsto A_nU + B_n$$

is an injective group homomorphism with

$$\bar{\tau}_n^F(Un_{E_{n-1}}\,F_{n-1}) \subset Un_{E_n}\,F_n\,.$$

For $U,V \in Un\,F_n$ we put $U \sim_1 V$ if $UV^*, U^*V \in Un\,E_n$. $\sim_1$ is an equivalence relation and $\sim_h$ implies $\sim_1$.

For $U,V \in Un\,F_{n-1}$,

$$\bar{\tau}_n^F U^* = A_nU^* + B_n = (\bar{\tau}_n^F U)^*\,,$$

$$(\bar{\tau}_n^F U)(\bar{\tau}_n^F V) = (A_nU + B_n)(A_nV + B_n) = A_nUV + B_n = \bar{\tau}_n^F(UV)\,,$$

$$(\bar{\tau}_n^F U)(\bar{\tau}_n^F U)^* = (\bar{\tau}_n^F U)^*(\bar{\tau}_n^F U) = A_n + B_n = 1_{F_n}\,,$$

i.e. $\bar{\tau}_n^F$ is well-defined and it is a group homomorphism. If $\bar{\tau}_n^F U = 1_{F_n}$ then

$$A_nU + B_n = \bar{\tau}_n^F U = 1_{F_n} = 1_E = A_n + B_n, \qquad A_nU = A_n\,,$$

so by Proposition 6.1.1 c), $U = 1_{F_{n-1}} = 1_E$ and $\bar{\tau}_n^F$ is injective.

The other assertions are obvious. ∎

DEFINITION 7.1.2 *Let F be a full E-C^*-algebra. We put for all $m,n \in \mathbb{N}$, $m < n$,*

$$\tau_{n,m}^F := \bar{\tau}_n^F \circ \bar{\tau}_{n-1}^F \circ \cdots \circ \bar{\tau}_{m+1}^F : Un\,F_m \longrightarrow Un\,F_n\,.$$

Then $\{(Un\,F_n)_{n\in\mathbb{N}},\ (\tau_{n,m})_{m,n\in\mathbb{N}}\}$ is an inductive system of groups with injective maps. We denote by $\{un F,\ (\tau_n^F)_{n\in\mathbb{N}}\}$ its inductive limit. τ_n^F is injective for every $n \in \mathbb{N}$, so $(\tau_n^F(Un\,F_n))_{n\in\mathbb{N}}$ is an increasing sequence of subgroups of $un F$, the union of which is $un F$. We put for every $n \in \mathbb{N}$ and $U \in Un\,F_n$,

$$Un\,F_{\leftarrow n} := \tau_n^F(Un\,F_n), \qquad U_{\leftarrow} := U_{\leftarrow n} := U_{\leftarrow n}^F := \tau_n^F U\,,$$

$$1_{\leftarrow n} := 1_{\leftarrow n}^F := \tau_n^F 1_{F_n}\,(= \tau_n^F 1_E)\,.$$

$(\tau_n^F(Un_{E_n}\,F_n))_{n\in\mathbb{N}}$ is an increasing sequence of subgroups of $un F$; we denote by $un_E\,F$ their union.

We often identify $Un\,F_n$ with $Un\,F_{\leftarrow n}$.

PROPOSITION 7.1.3 *For $m,n \in \mathbb{N}$, $m < n$, and $U \in Un\,F_m$,*

$$\tau^F_{n,m}U = \left(\prod_{i=m+1}^{n} A_i\right) U + \left(1_E - \prod_{i=m+1}^{n} A_i\right) .$$

We prove this identity by induction with respect to n. The identity holds for $n := m+1$. Assume it holds for $n-1 \geq m$. Then

$$\tau^F_{n,m}U = \bar{\tau}^F_n \tau^F_{n-1,m}U = A_n \tau^F_{n-1,m}U + B_n =$$

$$= A_n \left(\left(\prod_{i=m+1}^{n-1} A_i \right) U + \left(1_E - \prod_{i=m+1}^{n-1} A_i \right) \right) + B_n =$$

$$= \left(\prod_{i=m+1}^{n} A_i\right) U + \left(1_E - \prod_{i=m+1}^{n} A_i\right) . \qquad \blacksquare$$

PROPOSITION 7.1.4 *Let F be a full E-C*algebra.*

a) If $U,V \in Un\,F_{n-1}$ for some $n \in \mathbb{N}$ then

$$\bar{\tau}^F_n(UV) \sim_h \bar{\tau}^F_n(VU) , \qquad \bar{\tau}^F_n(UVU^*) \sim_h \bar{\tau}^F_n(V) .$$

b) $un_E\,F$ is a normal subgroup of $un\,F$ and $un\,F/un_E\,F$ is commutative.

c) For all $U,V \in un\,F$,

$$UV^* \in un_E\,F \Longleftrightarrow U^*V \in un_E\,F .$$

We put $U \sim_1 V$ if $UV^* \in un_E\,F$. $\sim_1$ is an equivalence relation.

a) By Proposition 6.2.5 a),b),

$$\bar{\tau}^F_n(UV) = A_nUV + B_n = (A_nU + B_n)(A_nV + B_n) \sim_h$$

$$\sim_h (A_nU + B_n)(A_n + B_nV) = A_nU + B_nV \sim_h A_nV + B_nU \sim_h \bar{\tau}^F_n(VU) .$$

It follows
$$\bar{\tau}_n^F(UVU^*) \sim_h \bar{\tau}_n^F(U^*UV) = \bar{\tau}_n^F(V)\,.$$

b) $un_E\,F$ is obviously a subgroup of $un\,F$. The other assertions follow from a).

c) Let $q : un\,F \to un\,F/un_E\,F$ be the quotient map. If $UV^* \in un_E\,F$ then by b),

$$q(UV^*) = q(U)q(V^*) = q(V^*)q(U) = q(V^*U)\,,$$

$$V^*U \in un_E\,F\,, \qquad\qquad U^*V = (V^*U)^* \in un_E\,F\,. \qquad \blacksquare$$

DEFINITION 7.1.5 *We denote for every E-C*-algebra F by $K_1(F)$ the additive group obtained from the commutative group $un\check{F}/un_E\,\check{F}$* (Proposition 7.1.4 b)) *by replacing the multiplication with the addition $\oplus$; by this the neutral element (which corresponds to 1_E) is denoted by* 0*. For every $U \in un\check{F}$ we denote by $[U]_1$ its equivalence class in $K_1(F)$.*

Remark. Let F be a full E-C*-algebra. By Proposition 4.1.2 d), $\check{F}$ is isomorphic to $E \times F$, so in this case we may define K_1 using F instead of $\check{F}$ (as we did for K_0).

PROPOSITION 7.1.6 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{M}_E$.*

a) For $m, n \in \mathbb{N}$, $m < n$, the diagram

$$\begin{array}{ccc} Un\,\check{F}_m & \xrightarrow{\tau_{n,m}^{\check{F}}} & Un\,\check{F}_n \\ {\scriptstyle\check{\varphi}_m}\downarrow & & \downarrow{\scriptstyle\check{\varphi}_n} \\ Un\,\check{G}_m & \xrightarrow[\tau_{n,m}^{\check{G}}]{} & Un\,\check{G}_n \end{array}$$

is commutative. Thus there is a unique group homomorphism

$$\check{\varphi}_{\leftarrow} : un\check{F} \longrightarrow un\check{G}$$

such that
$$\check{\varphi}_{\leftarrow} \circ \tau_n^{\check{F}} = \tau_n^{\check{G}} \circ \check{\varphi}_n$$

for every $n \in \mathbb{N}$.

b) $\varphi_{\leftarrow}(un_E\,\check{F}) \subset un_E\,\check{G}$*; if* φ *is surjective then* $\varphi_{\leftarrow}(un_E\,\check{F}) = un_E\,\check{G}$*.*

c) There is a unique group homomorphism

$$K_1(\varphi) : K_1(F) \longrightarrow K_1(G)$$

such that

$$K_1(\varphi)[U]_1 = [\check{\varphi}_{\leftarrow}U]_1$$

for every $U \in un\,\check{F}$*.*

d) $K_1(id_F) = id_{K_1(F)}$*.*

e) $K_1(\{0\}) = \{0\}$*.*

a) It is sufficient to prove the assertion for $n = m+1$. For $U \in Un\,\check{F}_m$,

$$\tau^{\check{G}}_{n,m}\check{\varphi}_m U = A_n(\check{\varphi}_m U) + B_n = \check{\varphi}_n(A_n U + B_n) = \check{\varphi}_n \tau^{\check{F}}_{n,m} U \ .$$

b) Since $\check{\varphi}_n(Un_{E_n}\,\check{F}_n) \subset Un_{E_n}\,\check{G}_n$ for every $n \in \mathbb{N}$, it follows $\varphi_{\leftarrow}(un_E\,\check{F}) \subset un_E\,\check{G}$. If φ is surjective then by [4] Lemma 2.1.7 (iii), we may replace the above inclusion relation by $=$.

c) follows from a) and b).

d) is obvious.

e) follows from $un\,E = un_E\,E$. ■

DEFINITION 7.1.7 *An E-C*-algebra F is called* **K-null** *if*

$$K_0(F) = K_1(F) = 0 \ .$$

Let $F \xrightarrow{\varphi} G$ *be a morphism in* $\mathfrak{M}_E$*. We say that* φ *is* **K-null** *if*

$$K_0(\varphi) = K_1(\varphi) = 0 \ .$$

We say that φ **factorizes through null** *if there are morphisms* $F \xrightarrow{\varphi'} H \xrightarrow{\varphi''} G$ *in* $\mathfrak{M}_E$ *such that* $\varphi = \varphi'' \circ \varphi'$ *and H is K-null.*

PROPOSITION 7.1.8

a) If $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ *are morphisms in* $\mathfrak{M}_E$ *then*

$$\check{\psi}_{\leftarrow} \circ \check{\varphi}_{\leftarrow} = (\check{\psi} \circ \check{\varphi})_{\leftarrow} = \left(\widecheck{\psi \circ \varphi}\right)_{\leftarrow}\ , \qquad K_1(\psi) \circ K_1(\varphi) = K_1(\psi \circ \varphi)\ .$$

b) If $\varphi = 0$ *then* $K_1(\varphi) = 0$.

c) **(Homotopy invariance of** K_1**)** *If* $\varphi, \psi : F \longrightarrow G$ *are homotopic morphisms in* $\mathfrak{M}_E$ *then*

$$K_1(\varphi) = K_1(\psi)\ .$$

d) **(Homotopy invariance of** K_1**)** *If* $F \xrightarrow{\varphi} G \xrightarrow{\psi} F$ *is a homotopy in* $\mathfrak{M}_E$ *then*

$$K_1(\varphi) : K_1(F) \longrightarrow K_1(G), \qquad K_1(\psi) : K_1(G) \longrightarrow K_1(F)$$

are isomorphisms and $K_1(\psi) = K_1(\varphi)^{-1}$.

e) If the E-C-algebra F is homotopic to* $\{0\}$ *then F is K-null.*

f) If a morphism in $\mathfrak{M}_E$ *factorizes through null then it is K-null.*

a) Since

$$\check{\psi}_n \circ \check{\varphi}_n = (\check{\psi} \circ \check{\varphi})_n = \left(\widecheck{\psi \circ \varphi}\right)_n$$

for every $n \in \mathbb{N}$ we get

$$\check{\psi}_{\leftarrow} \circ \check{\varphi}_{\leftarrow} = (\check{\psi} \circ \check{\varphi})_{\leftarrow} = \left(\widecheck{\psi \circ \varphi}\right)_{\leftarrow}\ .$$

For $U \in un\check{F}$, by Proposition 7.1.6 c),

$$K_1(\psi)K_1(\varphi)[U]_1 = K_1(\psi)[\check{\varphi}_{\leftarrow}U]_1 = [\check{\psi}_{\leftarrow}\check{\varphi}_{\leftarrow}U]_1 =$$

$$= [(\check{\psi} \circ \check{\varphi})_{\leftarrow}U]_1 = \left[\left(\widecheck{\psi \circ \varphi}\right)_{\leftarrow}U\right]_1 = K_1(\psi \circ \varphi)[U]_1\,,$$

so $K_1(\psi) \circ K_1(\varphi) = K_1(\psi \circ \varphi)$.

b) If we put $\vartheta : F \longrightarrow \{0\}$, $\iota : \{0\} \longrightarrow G$ then $\varphi = \iota \circ \vartheta$ and by a) and Proposition 7.1.6 e), $K_1(\varphi) = 0$.

c) Let

$$\phi_s : F \longrightarrow G, \qquad s \in [0,1]$$

be a pointwise continuous path of morphisms in $\mathfrak{M}_E$ with $\phi_0 = \varphi$ and $\phi_1 = \psi$. Let $n \in \mathbb{N}$. Then

$$(\check{\phi}_s)_n : \check{F}_n \longrightarrow \check{G}_n, \qquad s \in [0,1]$$

is a pointwise continuous path of E-C*-homomorphisms with $(\check{\phi}_0)_n = \check{\varphi}_n$ and $(\check{\phi}_1)_n = \check{\psi}_n$. For every $U \in Un\,\check{F}_n$, the map

$$\vartheta : [0,1] \longrightarrow Un\,\check{G}_n\,, \quad s \longmapsto (\check{\phi}_s)_n U$$

is continuous and $\vartheta(0) = \check{\varphi}_n U$, $\vartheta(1) = \check{\psi}_n U$, i.e. $\check{\varphi}_n U$ and $\check{\psi}_n U$ are homotopic in $Un\,\check{G}_n$. It follows

$$K_1(\varphi)[\tau_n^{\check{F}} U]_1 = K_1(\psi)[\tau_n^{\check{F}} U]_1\,,$$

which implies $K_1(\varphi) = K_1(\psi)$.

d) follows from c) and Proposition 7.1.6 d).

e) By d) and Proposition 7.1.6 e), $K_1(F) = \{0\}$. By the Homotopy invariance of K_0 (Theorem 6.2.11 e)), F is K-null.

f) follows immediately from a), e), and Corollary 6.2.3 a). ■

PROPOSITION 7.1.9 *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ *then*

$$K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \xrightarrow{K_1(\psi)} K_1(H)$$

is also exact.

Let $a \in Ker\,K_1(\psi)$ and let $U \in un\,\check{G}$ with $a = [U]_1$. By Proposition 7.1.6 c),

$$0 = K_1(\psi)a = [\check{\psi}_{\leftarrow} U]_1\,, \qquad \check{\psi}_{\leftarrow} U \in un_E\,\check{H}\,.$$

By Proposition 7.1.6 b), there is a $V \in un_E \check{G}$ with $\check{\psi}_{\leftarrow} V = \check{\psi}_{\leftarrow} U$. We put $W := UV^*$. By Proposition 7.1.4 c), $[W]_1 = a$ and so

$$\check{\psi}_{\leftarrow} W = (\check{\psi}_{\leftarrow} U)(\check{\psi}_{\leftarrow} V)^* = 1_E \ .$$

W has the form

$$W = \sum_{t \in T_n} ((\alpha_t, X_t) \otimes id_K) V_t^{\check{G}}$$

for some $n \in \mathbb{N}$, where $(\alpha_t, X_t) \in E \times G$ for every $t \in T_n$. We get

$$1_E = \check{\psi}_n W = \sum_{t \in T_n} ((\alpha_t, \psi X_t) \otimes id_K) V_t^{\check{H}}$$

and so by [2] Theorem 2.1.9 a), $\psi X_t = 0$ for every $t \in T_n$. For every $t \in T_n$, let $Y_t \in F$ with $\varphi Y_t = X_t$ and put

$$W' := \sum_{t \in T_n} ((\alpha_t, Y_t) \otimes id_K) V_t^{\check{F}} \ .$$

Since $\check{\varphi} : \check{F} \longrightarrow \check{G}$ is an embedding, $W' \in Un\, \check{F}_{\leftarrow n}$ and by Proposition 7.1.6 c),

$$K_1(\varphi)[W']_1 = [\check{\varphi}_n W']_1 = [W]_1 = a \ .$$

Thus $Ker\, K_1(\psi) \subset Im\, K_1(\varphi)$.

Let now $U \in un \check{F}_{\leftarrow}$. By Proposition 7.1.8 a),b),

$$K_1(\psi) K_1(\varphi)[U]_1 = K_1(\psi \circ \varphi)[U]_1 = K_1(0)[U]_1 = 0$$

so $Im\, K_1(\varphi) \subset Ker\, K_1(\psi)$. ■

PROPOSITION 7.1.10 *The following are equivalent for every full E-C*-algebra F.*

a) $K_1(F) = \{0\}$.

b) For every $n \in \mathbb{N}$ *and* $U \in Un\, F_n$ *there is an* $m \in \mathbb{N}$, $m > n$, *with* $\tau^F_{m,n} U \sim_h 1_E$ *in* $Un\, F_m$.

$a \Rightarrow b$ Since

$$(1_E, U) \in Un\, E_n \times Un\, F_n = Un\, (E_n \times F_n) = Un\, (E \times F)_n \,,$$

it follows from Proposition 4.1.2 d), $(1_E, U - 1_E) \in Un\,\check{F}_n$. By a), there is an $m \in \mathbb{N}$, $m > n$, with

$$U_0 := (1_E, \tau^F_{m,n}U - 1_E) = \tau^{\check{F}}_{m,n}(1_E, U - 1_E) \in Un_{E_m}\check{F}_m\,.$$

Thus there is a continuous map

$$[0,1] \longrightarrow Un\,\check{F}_m, \quad s \longmapsto U_s$$

with $U_1 \in Un\,E_m\,(\subset Un\,\check{F}_m)$. We put

$$U'_s := U_s(\sigma^F_m U_s)^*\,(\in Un\,\check{F}_m)$$

for every $s \in [0,1]$. Then the map

$$[0,1] \longrightarrow Un\,\check{F}_m, \quad s \longmapsto U'_s$$

is continuous and $U'_0 = U_0$, $U'_1 = 1_E$. Let

$$\varphi : \check{F} \longrightarrow E \times F\,, \quad (\alpha, x) \longmapsto (\alpha, x + \alpha)$$

be the E-C*-isomorphism of Proposition 4.1.2 d). Then

$$U'' : [0,1] \longrightarrow Un\,E_n \times Un\,F_n\,, \quad s \longmapsto \varphi_m U'_s$$

is continuous and

$$U''_0 = \varphi_m U'_0 = (1_E, \tau^F_{m,n}U)\,, \qquad U''_1 = \varphi_m U'_1 = (1_E, 1_E)\,.$$

Thus $\tau^F_{m,n}U \sim_h 1_E$ in $Un\,F_m$.

$b \Rightarrow a$ Let $a \in K_1(F)$. There are $n \in \mathbb{N}$ and $U \in Un\,\check{F}_n$ with $a = [U]_1$. Since $U(\sigma^F_n U)^* \sim_1 U$, we may assume $U = U(\sigma^F_n U)^*$, i.e. $\sigma^F_n U = 1_E$. Thus there is a unique $X \in F_n$ with $\iota^F_n X = U - 1_E$. Then

$$U' := X + 1_E \in Un\,F_n\,.$$

By b), there is an $m \in \mathbb{N}$, $m > n$, with $\tau^F_{m,n}U' \sim_h 1_E$. By Proposition 4.1.2 d),

$$U = (1_E, X) = (1_E, U' - 1_E)\,, \qquad \tau^{\check{F}}_{m,n}U = (1_E, \tau^F_{m,n}U' - 1_E) \sim_h (1_E, 0)\,,$$

i.e. $a = [U]_1 = 0$. ■

COROLLARY 7.1.11 *If F is a finite-dimensional full E-C*-algebra then $K_1(F) = \{0\}$.*

For every $n \in \mathbb{N}$, F_n is finite-dimensional and so there is a finite family $(k_i)_{i\in I}$ in $\mathbb{N}$ such that $F_n \approx \prod_{i\in I} \mathbb{C}_{k_i,k_i}$. Thus every $U \in Un\,F_n$ is homotopic to 1_E in $Un\,F_n$. By Proposition 7.1.10 $b \Rightarrow a$, $K_1(F) = \{0\}$. ■

COROLLARY 7.1.12 *If the spectrum of E is totally disconnected (this happens e.g. if E is a W*-algebra* ([1] Corollary 4.4.1.10)*) then $Un\,E_n = Un_0\,E_n$ for every $n \in \mathbb{N}$ and so $K_1(E) = \{0\}$.*

Let Ω be the spectrum of E and let $U \in Un\,E_n$. U has the form

$$U = \sum_{t\in T_n} (U_t \otimes id_K)V_t\,,$$

with $U_t \in E$ for every $t \in T_n$. We put

$$U(\omega) := \sum_{t\in T_n} (U_t(\omega) \otimes id_K)V_t$$

for every $\omega \in \Omega$ and denote by $\sigma(U(\omega))$ its spectrum, which is finite. Let $\omega_0 \in \Omega$ and let $\theta_0 \in [0,2\pi[$ such that $e^{i\theta_0} \notin \sigma(U(\omega_0))$. By [1] Corollary 2.2.5.2, there is o clopen neighborhood Ω_0 of ω_0 such that $e^{i\theta_0}$ does not belong to the spectrum of $U(\omega)$ for all $\omega \in \Omega_0$. Assume for a moment $\Omega_0 = \Omega$ and put for every $s \in [0,1]$,

$$h_s : \mathbb{T}\setminus\{\alpha\} \longrightarrow \mathbb{T}, \quad e^{i\vartheta} \longmapsto e^{i\vartheta s}, \qquad W_s := h_s(U)\,,$$

where $\vartheta \in]\vartheta_0 - 2\pi, \vartheta_0[$. Then

$$[0,1] \longrightarrow Un\,E_n, \quad s \longmapsto W_s$$

is a continuous path in $Un\,E_n$ ([1] Corollaries 4.1.2.13 and 4.1.3.5) with $W_1 = U$ and $W_0 = 1_E$. Thus $U \in Un_0\,E_n$.

Since Ω is the union of a finite family of pairwise disjoint clopen sets of the above form Ω_0, $U \in Un_0\,E_n$.

By Proposition 7.1.10 $b \Rightarrow a$, $K_1(E) = \{0\}$. ■

7.2 The Index Map

Throughout this section

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

denotes an exact sequence in $\mathfrak{M}_E$ and $n \in \mathbb{N}$.

PROPOSITION 7.2.1 *Let $U \in Un\,\check{H}_{n-1}$.*

a) There are $V \in Un\,\check{G}_n$ and $P \in Pr\,\check{F}_n$ such that

$$\check{\psi}_n V = A_n U + B_n U^*, \qquad \check{\varphi}_n P = V A_n V^* \,.$$

b) If $W \in Un\,\check{G}_n$ and $Q \in Pr\,\check{F}_n$ such that

$$\check{\psi}_n W = A_n U + B_n U^*, \qquad \check{\varphi}_n Q = W A_n W^*$$

then $\sigma_n^F Q = A_n$ and $P \sim_0 Q$.

c) Let $U_0 \in Un\,\check{H}_{n-1}$, $V_0 \in Un\,\check{G}_n$, and $P_0 \in Pr\,\check{F}_n$ with

$$U_0 \sim_1 U, \qquad \check{\psi}_n V_0 = A_n U_0 + B_n U_0^*, \qquad \check{\varphi}_n P_0 = V_0 A_n V_0^* \,.$$

Then $P_0 \sim_0 P$.

d) If $U \in Un_{E_{n-1}}\,\check{H}_{n-1}$ then $P \sim_0 A_n$.

a) By Proposition 6.2.5 d), $A_n U + B_n U^* \in Un_0\,\check{H}_n$ so by [4] Lemma 2.1.7 (i) (and [2] Theorem 2.1.9 a)), there is a $V \in Un_0\,\check{G}_n$ with $\check{\psi}_n V = A_n U + B_n U^*$. We have

$$\check{\psi}_n (V A_n V^*) = (A_n U + B_n U^*) A_n (A_n U^* + B_n U) = A_n \,,$$

$$\sigma_n^H \check{\psi}_n (V A_n V^*) = \sigma_n^H A_n = A_n = \check{\psi}_n (V A_n V^*) \,,$$

so by Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $P \in Pr\,\check{F}_n$ with $\check{\varphi}_n P = V A_n V^*$.

b) Since $\pi^F = \pi^H \circ \check{\psi} \circ \check{\varphi}$, we have

$$\pi_n^F Q = \pi_n^H \check{\psi}_n \check{\varphi}_n Q = \pi_n^H \check{\psi}_n (W A_n W^*) =$$

$$= \pi_n^H((A_nU + B_nU^*)A_n(A_nU^* + B_nU)) = \pi_n^H A_n = A_n\,,$$

$\sigma_n^F Q = A_n$. Since

$$\check{\psi}_n(WV^*) = (A_nU + B_nU^*)(A_nU^* + B_nU) = A_n + B_n = 1_E = \sigma_n^H \check{\psi}_n(WV^*)\,,$$

by Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $Z \in Un\,\check{F}_n$ with $\check{\varphi}_n Z = WV^*$. Then

$$\check{\varphi}_n(ZPZ^*) = (WV^*)(VA_nV^*)(VW^*) = WA_nW^* = \check{\varphi}_n Q\,,$$

$$ZPZ^* = Q, \qquad\qquad P \sim_0 Q\,.$$

c) By Proposition 7.1.4 c), $U^*U_0, UU_0^* \in Un_{E_{n-1}}\,\check{H}_{n-1}$ so by [4] Lemma 2.1.7 (iii), there are $X, Y \in Un\,\check{G}_{n-1}$ such that

$$\check{\psi}_{n-1}X = U^*U_0, \qquad\qquad \check{\psi}_{n-1}Y = UU_0^*\,.$$

We put

$$Z := V(A_nX + B_nY)\,.$$

By Proposition 6.2.5 c), $Z \in Un\,\check{G}_n$. We have

$$\check{\psi}_n Z = (A_nU + B_nU^*)(A_nU^*U_0 + B_nUU_0^*) = A_nU_0 + B_nU_0^*\,,$$

$$\check{\psi}_n(ZA_nZ^*) = (A_nU_0 + B_nU_0^*)A_n(A_nU_0^* + B_nU_0) = A_n = \sigma_n^H \check{\psi}_n(ZA_nZ^*)\,.$$

By Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $Q \in Pr\,\check{F}_n$ with $\check{\varphi}_n Q = ZA_nZ^*$. By b), $Q \sim_0 P_0$. From

$$\check{\varphi}_n Q = ZA_nZ^* = V(A_nX + B_nY)A_n(A_nX^* + B_nY^*)V^* = VA_nV^* = \check{\varphi}_n P$$

it follows $P_0 \sim_0 Q = P$ (by [2] Theorem 2.1.9 a)).

d) By c), we may take $U = 1_E$. Further we may take $W = 1_E$ and $Q = A_n$ in b), so $P \sim A_n$. ■

PROPOSITION 7.2.2 *For every $i \in \{1,2\}$ let $U_i \in Un\,\check{H}_{n-1}$, $V_i \in Un\,\check{G}_n$, and $P_i \in Pr\,\check{F}_n$ such that*

$$\check{\psi}_n V_i = A_nU_i + B_nU_i^*, \qquad\qquad \check{\varphi}_n P_i = V_iA_nV_i^*\,.$$

Put

$$X := A_{n+1}A_n + C_{n+1}^*C_n + C_{n+1}C_n^* + B_{n+1}B_n, \qquad U := A_nU_1 + B_nU_2\,,$$

$$V := X(A_{n+1}V_1 + B_{n+1}V_2)X, \qquad\qquad P := X(A_{n+1}P_1 + B_{n+1}P_2)X\,,$$

a) $X \in Un_0\, E_{n+1},\ U \in Un\, \check{H}_n,\ V \in Un\, \check{G}_{n+1},\ P \in Pr\, \check{F}_{n+1}.$

b) $\check{\psi}_{n+1} V = A_{n+1} U + B_{n+1} U^*,\ \check{\varphi}_{n+1} P = V A_{n+1} V^*.$

a) We have

$$X^2 = A_{n+1}A_n + A_{n+1}B_n + B_{n+1}A_n + B_{n+1}B_n = 1_E \ .$$

Since X is selfadjoint it follows $X \in Un_0\, E_{n+1}$ ([4] Lemma 2.1.3 (ii)) and so $P \in Pr\, \check{F}_{n+1}$. By Proposition 6.2.5 c), $U \in Un\, \check{H}_n$ and $V \in Un\, \check{G}_{n+1}$.

b) We have

$$XA_{n+1}X = (A_{n+1}A_n + C_{n+1}C_n^*)X = A_{n+1}A_n + B_{n+1}A_n = A_n \,,$$

$$XB_{n+1}X = (C_{n+1}^* C_n + B_{n+1}B_n)X = A_{n+1}B_n + B_{n+1}B_n = B_n \,,$$

$$XA_nX = A_{n+1}, \qquad XB_nX = B_{n+1} \,,$$

$$XA_{n+1}A_nX = A_{n+1}A_n, \qquad XA_{n+1}B_nX = B_{n+1}A_n \,,$$

$$XB_{n+1}A_nX = A_{n+1}B_n, \qquad XB_{n+1}B_nX = B_{n+1}B_n \,,$$

$$\check{\psi}_{n+1} V = X(A_{n+1}(A_nU_1 + B_nU_1^*) + B_{n+1}(A_nU_2 + B_nU_2^*))X =$$

$$= A_{n+1}A_nU_1 + B_{n+1}A_nU_1^* + A_{n+1}B_nU_2 + B_{n+1}B_nU_2^* = A_{n+1}U + B_{n+1}U^* \,,$$

$$VA_{n+1}V^* = X(A_{n+1}V_1 + B_{n+1}V_2)XA_{n+1}X((A_{n+1}V_1^* + B_{n+1}V_2^*)X =$$

$$= X(A_{n+1}V_1 + B_{n+1}V_2)A_n(A_{n+1}V_1^* + B_{n+1}V_2^*)X =$$

$$= X(A_{n+1}V_1A_nA_{n+1}V_1^* + B_{n+1}V_2A_nB_{n+1}V_2^*)X =$$

$$= X(A_{n+1}V_1A_nV_1^* + B_{n+1}V_2A_nV_2^*)X =$$

$$= X(A_{n+1}\check{\varphi}_nP_1 + B_{n+1}\check{\varphi}_nP_2)X =$$

$$= \check{\varphi}_{n+1}(X(A_{n+1}P_1 + B_{n+1}P_2)X) = \check{\varphi}_{n+1}P \ . \qquad \blacksquare$$

COROLLARY 7.2.3 *There is a unique group homomorphism, called* **the index map***,*

$$\delta_1 : K_1(H) \longrightarrow K_0(F)$$

such that

$$\delta_1[U]_1 = [P]_0 - [\sigma^F_{\rightarrow} P]_0$$

for every $U \in un\check{H}$, where P satisfies the conditions of Proposition 7.2.1 a).

By Proposition 7.2.1 a),b), the map

$$\nu_n : Un\,\check{H}_{n-1} \longrightarrow K_0(F)\,, \quad U \longmapsto [P]_0 - [\sigma_n^F P]_0$$

is well-defined for every $n \in \mathbb{N}$, where P is associated to U as in Proposition 7.2.1 a). By Proposition 7.2.1 c), $\nu_n U = \nu_n U_0$ for all $U, U_0 \in Un\,\check{H}_{n-1}$ with $U \sim_1 U_0$. With the notation of Proposition 7.2.2,

$$\nu_{n+1}(A_n U_1 + B_n U_2) = \nu_{n+1} U = [P]_0 - [\sigma_{n+1}^F P]_0 =$$

$$= [A_{n+1} P_1 + B_{n+1} P_2]_0 - [\sigma_{n+1}^F (A_{n+1} P_1 + B_{n+1} P_2)]_0 =$$

$$= [P_1]_0 + [P_2]_0 - [\sigma_n^F P_1]_0 - [\sigma_n^F P_2]_0 = \nu_n U_1 + \nu_n U_2\,.$$

Thus by Proposition 7.2.1 d) (and Proposition 7.2.2), for $U \in Un\,\check{H}_{n-1}$,

$$\nu_{n+1}(\bar{\tau}_n^{\check{H}} U) = \nu_{n+1}(A_n U + B_n) = \nu_n U + \nu_n 1_E = \nu_n U\,.$$

Hence the map

$$\nu : un\check{H} \longrightarrow K_0(F)\,, \quad U \longmapsto \nu_n U$$

is well-defined, where $U \in Un\,\check{H}_{n-1}$ for some $n \in \mathbb{N}$. By Proposition 7.2.1 d), again, ν induces a map $\delta_1 : K_1(H) \longrightarrow K_0(F)$, which is additive by the above considerations. The uniqueness follows from the fact that the map $[\cdot]_1 : un\check{H} \longrightarrow K_1(H)$ is surjective. ■

PROPOSITION 7.2.4 *Let*

$$0 \longrightarrow F' \xrightarrow{\varphi'} G' \xrightarrow{\psi'} H' \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$ *and* δ_1' *its associated index map. If the diagram in* $\mathfrak{M}_E$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & H & \longrightarrow & 0 \\
 & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & F' & \xrightarrow[\varphi']{} & G' & \xrightarrow[\psi']{} & H' & \longrightarrow & 0
\end{array}$$

is commutative then the diagram

$$\begin{array}{ccc}
K_1(H) & \xrightarrow{\delta_1} & K_0(F) \\
{\scriptstyle K_1(\beta)}\downarrow & & \downarrow{\scriptstyle K_0(\gamma)} \\
K_1(H') & \xrightarrow[\delta_1']{} & K_0(F')
\end{array}$$

is also commutative.

Let $U \in Un\,\check{H}_{n-1}$, $V \in Un\,\check{G}_n$, and $P \in Pr\,\check{F}_n$ with

$$\check{\psi}_n V = A_n U + B_n U^*, \qquad \check{\varphi}_n P = V A_n V^* \,.$$

Put

$$V' := \check{\alpha}_n V \in Un\,\check{G}'_n, \qquad P' := \check{\gamma}_n P \in Pr\,\check{F}'_n \,.$$

Then

$$\check{\psi}'_n V' = \check{\psi}'_n \check{\alpha}_n V = \check{\beta}_n \check{\psi}_n V = A_n \check{\beta}_{n-1} U + B_n \check{\beta}_{n-1} U^* \,,$$

$$\check{\varphi}'_n P' = \check{\varphi}'_n \check{\gamma}_n P = \check{\alpha}_n \check{\varphi}_n P = \check{\alpha}_n (V A_n V^*) = V' A_n V'^* \,.$$

By Corollary 7.2.3 for δ'_1, Proposition 7.1.6 c), and Proposition 6.2.2 c),

$$\delta'_1 K_1(\beta)[U]_1 = \delta'_1[\check{\beta}_{n-1} U]_1 = [P']_0 - [\sigma_n^{F'} P']_0 = [\check{\gamma}_n P]_0 - [\sigma_n^{F'} \check{\gamma}_n P]_0 =$$

$$= [\check{\gamma}_n P]_0 - [\check{\gamma}_n \sigma_n^F P]_0 = K_0(\gamma)([P]_0 - [\sigma_n^F P]_0) = K_0(\gamma)\delta_1[U]_1 \,. \qquad \blacksquare$$

PROPOSITION 7.2.5

a) $\delta_1 \circ K_1(\psi) = 0$.

b) $K_0(\varphi) \circ \delta_1 = 0$.

a) Let $U \in Un\,\check{G}_{n-1}$ and put

$$V := \bar{\tau}_n^{\check{G}} U = A_n U + B_n \in Un\,\check{G}_n \,.$$

Then

$$\check{\psi}_n V = A_n(\check{\psi}_{n-1} U) + B_n \,,$$

$$(\check{\psi}_n V) A_n (\check{\psi}_n V)^* = (A_n(\check{\psi}_{n-1} U) + B_n) A_n (A_n(\check{\psi}_{n-1} U)^* + B_n) = A_n \,,$$

so (by Proposition 7.1.6 c))

$$\delta_1 K_1(\psi)[U]_1 = \delta_1[\check{\psi}_{n-1} U]_1 = [A_n]_0 - [\sigma_n^F A_n]_0 = 0 \,.$$

b) Let $U \in Un\,\check{H}_{n-1}$, $V \in Un\,\check{G}_n$, and $P \in Pr\,\check{F}_n$ with

$$\check{\psi}_n V = A_n U + B_n U^*, \qquad \check{\varphi}_n P = V A_n V^* \,.$$

By Proposition 6.2.2 c) (since $\check{\varphi}\circ\sigma^F = \sigma^G\circ\check{\varphi}$),

$$K_0(\varphi)\delta_1[U]_1 = K_0(\varphi)([P]_0 - [\sigma_n^F P]_0) =$$

$$= [\check{\varphi}_n P]_0 - [\check{\varphi}_n \sigma_n^F P]_0 = [\check{\varphi}_n P]_0 - [\sigma_n^G \check{\varphi}_n P]_0 =$$

$$= [VA_nV^*]_0 - [(\sigma_n^G V)A_n(\sigma_n^G V)^*]_0 = [A_n]_0 - [A_n]_0 = 0\,.$$ ■

PROPOSITION 7.2.6 *Let $U \in Un\,\check{H}_{n-1}$. There are $V \in \check{G}_n$ and $P,Q \in Pr\,\check{F}_n$ such that*

$$V^*V \in Pr\,\check{G}_n, \qquad \check{\psi}_n V = A_n U\,,$$

$$\check{\varphi}_n P = 1_E - V^*V, \qquad \check{\varphi}_n Q = 1_E - VV^*, \qquad \delta_1[U]_1 = [P]_0 - [Q]_0\,.$$

By Proposition 6.2.5 d), $A_nU + B_nU^* \in Un_0\,\check{H}_n$. Since $\check{\psi}_n$ is surjective, by [4] Lemma 2.1.7 (i), there is a $V_0 \in Un\,\check{G}_n$ with $\check{\psi}_n V_0 = A_nU + B_nU^*$. Put $V := V_0A_n \in \check{G}_n$. Then

$$V^*V = A_nV_0^*V_0A_n = A_n \in Pr\,\check{G}_n$$

and

$$\check{\psi}_n V = (\check{\psi}_n V_0)A_n = (A_nU + B_nU^*)A_n = A_nU\,.$$

We have

$$\check{\psi}_n(1_E - V^*V) = 1_E - A_n = B_n = \check{\psi}_n(1_E - VV^*)\,.$$

By Proposition 6.2.8 $b_2 \Rightarrow b_1$, there are $P,Q \in Pr\,\check{F}_n$ with

$$\check{\varphi}_n P = 1_E - V^*V, \qquad \check{\varphi}_n Q = 1_E - VV^*\,.$$

Put

$$W := A_{n+1}V + C_{n+1}(1_E - V^*V) + C_{n+1}^*(1_E - VV^*) + B_{n+1}V^* \in \check{G}_{n+1}\,,$$

$$Z := A_n + (C_{n+1} + C_{n+1}^*)B_n \in E_{n+1}\,.$$

Since $VV^*V = V$, $V^*VV^* = V^*$, and

$$W^* = A_{n+1}V^* + C_{n+1}^*(1_E - V^*V) + C_{n+1}(1_E - VV^*) + B_{n+1}V\,,$$

we get

$$WW^* = A_{n+1}VV^* + B_{n+1}(1_E - V^*V) + A_{n+1}(1_E - VV^*) + B_{n+1}V^*V =$$

$$= A_{n+1} + B_{n+1} = 1_E \,,$$

$$W^*W = A_{n+1}V^*V + A_{n+1}(1_E - V^*V) + B_{n+1}(1_E - VV^*) + B_{n+1}VV^* =$$

$$= A_{n+1} + B_{n+1} = 1_E \ .$$

By Proposition 6.2.5 a),

$$Z^2 = A_n + B_n = 1_E$$

so $W \in Un\, \check{G}_{n+1}$, $Z \in Un\, E_{n+1}$, and $ZW \in Un\, \check{G}_{n+1}$. By the above and Proposition 6.2.5 a),

$$\check{\psi}_{n+1}W = A_{n+1}A_nU + (C_{n+1} + C^*_{n+1})B_n + B_{n+1}A_nU^* \,,$$

$$\check{\psi}_{n+1}(ZW) = Z\check{\psi}_{n+1}W =$$

$$= (A_n + (C_{n+1} + C^*_{n+1})B_n)(A_{n+1}A_nU + (C_{n+1} + C^*_{n+1})B_n + B_{n+1}A_nU^*) =$$

$$= A_{n+1}A_nU + B_{n+1}A_nU^* + B_n = A_{n+1}A_nU + B_{n+1}A_nU^* + (A_{n+1} + B_{n+1})B_n =$$

$$= A_{n+1}(A_nU + B_n) + B_{n+1}(A_nU^* + B_n) \ .$$

We put

$$R := A_{n+1}(1_E - Q) + B_{n+1}P \in Pr\, \check{F}_{n+1} \ .$$

Using again $VV^*V = V$ and $V^*VV^* = V^*$,

$$\check{\varphi}_{n+1}R = A_{n+1}VV^* + B_{n+1}(1_E - V^*V) \,,$$

$$WA_{n+1} = A_{n+1}V + C_{n+1}(1_E - V^*V) \,,$$

$$WA_{n+1}W^* = A_{n+1}VV^* + B_{n+1}(1_E - V^*V) = \check{\varphi}_{n+1}R \,,$$

$$ZWA_{n+1}W^*Z = Z(\check{\varphi}_{n+1}R)Z = \check{\varphi}_{n+1}(ZRZ) \ .$$

Since $ZRZ \sim_0 R$ and $U \sim_1 A_nU + B_n$, by the definition of δ_1,

$$\delta_1[U]_1 = \delta_1[A_nU + B_n]_1 = [R]_0 - [\sigma^F_{n+1}R]_0 \ .$$

Since $\pi^H \circ \check{\psi} \circ \check{\varphi} = \pi^F$, by the above,

$$\pi^F_n P = \pi^H_n \check{\psi}_n \check{\varphi}_n P = \pi^H_n \check{\psi}_n (1_E - V^*V) = \pi^H_n B_n = B_n = \pi^F_n Q \ .$$

Thus by Proposition 6.1.3 (and Proposition 7.2.1 b)),

$$\sigma^F_{n+1}R = A_{n+1}(1_E - B_n) + B_{n+1}B_n \sim_0 A_{n+1}B_n + A_{n+1}A_n =$$

$$= A_{n+1} = \bar{\rho}^{\check{F}}_{n+1} 1_E \sim_0 1_E$$

and we get

$$[R]_0 = [1_E - Q]_0 + [P]_0 = [1_E]_0 + [P]_0 - [Q]_0\,,$$

$$\delta_1[U]_1 = [1_E]_0 + [P]_0 - [Q]_0 - [1_E]_0 = [P]_0 - [Q]_0\,. \qquad \blacksquare$$

PROPOSITION 7.2.7 *Ker* $\delta_1 \subset$ *Im* $K_1(\psi)$.

Let $a \in Ker\ \delta_1$ and let $U \in Un\ \check{H}_{n-1}$ with $a = [U]_1$. By Proposition 7.2.6, there are $V \in \check{G}_n$ and $P, Q \in Pr\ \check{F}_n$ such that $V^*V \in Pr\ \check{G}_n$, $\check{\psi}_n V = A_n U$,

$$\check{\varphi}_n P = 1_E - V^*V, \qquad \check{\varphi}_n Q = 1_E - VV^*, \qquad \delta_1[U]_1 = [P]_0 - [Q]_0\,.$$

Then $[P]_0 = [Q]_0$. By Corollary 6.1.6 a⇒c, there is an $m \in \mathbb{N}$, $m > n+1$, and an $X \in \check{F}_m$ such that

$$X^*X = \left(\prod_{i=n+1}^{m} A_i\right) P + \left(1_E - \prod_{i=n+1}^{m} A_i\right)\,,$$

$$XX^* = \left(\prod_{i=n+1}^{m} A_i\right) Q + \left(1_E - \prod_{i=n+1}^{m} A_i\right)\,.$$

Put $W := \check{\varphi}_m X$. Then

$$W^*W = \check{\varphi}_m(X^*X) = \left(\prod_{i=n+1}^{m} A_i\right)(1_E - V^*V) + \left(1_E - \prod_{i=n+1}^{m} A_i\right) =$$

$$= 1_E - \left(\prod_{i=n+1}^{m} A_i\right) V^*V\,,$$

$$WW^* = 1_E - \left(\prod_{i=n+1}^{m} A_i\right) VV^*\,,$$

$$\left(\prod_{i=n+1}^{m} A_i\right) VV^*WW^* = \left(\prod_{i=n+1}^{m} A_i\right) V^*VW^*W = 0\,,$$

$$\left(\prod_{i=n+1}^{m} A_i\right) V^*W = \left(\prod_{i=n+1}^{m} A_i\right) VW^* = 0\,,$$

$$\left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right)^* \left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right) =$$

$$= \left(\prod_{i=n+1}^{m} A_i\right) V^*V + W^*W = 1_E\,,$$

$$\left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right)\left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right)^* =$$

$$= \left(\prod_{i=n+1}^{m} A_i\right) VV^* + WW^* = 1_E\,,$$

$$\left(\prod_{i=n+1}^{m} A_i\right) V + W \in Un\,\check{G}_m\,.$$

From

$$\check{\psi}_m(W^*W) = 1_E - \left(\prod_{i=n+1}^{m} A_i\right)\check{\psi}_m(V^*V) =$$

$$= 1_E - \left(\prod_{i=n+1}^{m} A_i\right) A_n = \check{\psi}_m(WW^*)\,,$$

since $\check{\psi}_m W = \check{\psi}_m \check{\varphi}_m X \in E_m$, it follows

$$\check{\psi}_m W + \left(\prod_{i=n}^{m} A_i\right) \in Un\,E_m\,.$$

By the above,

$$\left(\prod_{i=n}^{m} A_i\right) U\,\check{\psi}_m W^* = \left(\prod_{i=n+1}^{m} A_i\right)(\check{\psi}_m V)(\check{\psi}_m W^*) =$$

$$= \check{\psi}_m\left(\left(\prod_{i=n+1}^{m} A_i\right) VW^*\right) = 0\,,$$

$$(\check{\psi}_m W)^*(\check{\psi}_m W)\left(\prod_{i=n}^{m} A_i\right) = 0\,, \qquad (\check{\psi}_m W)\left(\prod_{i=n}^{m} A_i\right) = 0\,,$$

$$\check{\psi}_m\left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right) = \left(\prod_{i=n}^{m} A_i\right) U + \check{\psi}_m W \sim_1$$

$$\sim_1 \left(\left(\prod_{i=n}^{m} A_i\right) U + \check{\psi}_m W\right)\left(\left(\prod_{i=n}^{m} A_i\right) + \check{\psi}_m W^*\right) =$$

$$= \left(\left(\prod_{i=n}^{m} A_i\right) U + \left(1_E - \prod_{i=n}^{m} A_i\right)\right)\,.$$

By Proposition 7.1.3 and Proposition 7.1.6 c),

$$a = [U]_1 = \left[\left(\prod_{i=n}^{m} A_i\right) U + \left(1_E - \prod_{i=n}^{m} A_i\right)\right]_1 =$$

$$= \left[\check{\psi}_m \left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right)\right]_1 =$$

$$= K_1(\psi)\left[\left(\prod_{i=n+1}^{m} A_i\right) V + W\right]_1 \in Im\, K_1(\psi)\,. \qquad \blacksquare$$

PROPOSITION 7.2.8 *$Ker\, K_0(\varphi) \subset Im\, \delta_1$.*

Let $a \in Ker\, K_0(\varphi)$. By Proposition 6.2.4, there is a $P \in Pr\, \check{F}_{\rightarrow}$ with

$$a = [P]_0 - [\sigma^F_{\rightarrow} P]_0\,.$$

By Proposition 6.2.2 c),

$$0 = K_0(\varphi)a = [\check{\varphi}_{\rightarrow} P]_0 - [\check{\varphi}_{\rightarrow} \sigma^F_{\rightarrow} P]_0\,.$$

Let $n \in \mathbb{N}$ such that $P \in Pr\, \check{F}_{\rightarrow n}$. Then $[\check{\varphi}_{\rightarrow n} P]_0 = [\check{\varphi}_{\rightarrow n} \sigma^F_{\rightarrow n} P]_0$. By Corollary 6.1.6 a$\Rightarrow$c, there is an $m \in \mathbb{N}$, $m > n+1$, such that

$$\check{\varphi}_{\rightarrow n} P + (B_m)_{\rightarrow} \sim_0 \check{\varphi}_{\rightarrow n} \sigma^F_{\rightarrow n} P + (B_m)_{\rightarrow}\,.$$

Put

$$Q := P + (B_m)_{\rightarrow} \in Pr\, \check{F}_{\rightarrow m}\,.$$

Then

$$a = [Q]_0 - [\sigma^F_{\rightarrow} Q]_0, \qquad \check{\varphi}_{\rightarrow m} Q \sim_0 \check{\varphi}_{\rightarrow m} \sigma^F_{\rightarrow m} Q = \sigma^F_{\rightarrow m} Q\,.$$

By Proposition 6.2.6, there are $k \in \mathbb{N}$, $k \geq m+2$, and $W \in Un\, \check{G}_{\rightarrow k}$ with

$$W(\check{\varphi}_{\rightarrow m} Q)W^* = \sigma^F_{\rightarrow m} Q\,.$$

It follows

$$(\sigma^F_{\rightarrow m} Q)W = W(\check{\varphi}_{\rightarrow m} Q)W^* W = W(\check{\varphi}_{\rightarrow m} Q)\,,$$

$$(\check{\psi}_{\rightarrow k} W)(\sigma^F_{\rightarrow k} Q) = (\check{\psi}_{\rightarrow k} W)(\check{\psi}_{\rightarrow k} \check{\varphi}_{\rightarrow k} Q) = \check{\psi}_{\rightarrow k}(W \check{\varphi}_{\rightarrow k} Q) =$$

$$= \check{\psi}_{\to k}((\sigma^F_{\to k}Q)W) = (\sigma^F_{\to k}Q)(\check{\psi}_{\to k}W)\ .$$

Put

$$U := (\check{\psi}_{\to k}W)(1_E - \sigma^F_{\to k}Q) + \sigma^F_{\to k}Q \in \check{H}_{\to k}\ .$$

Then

$$UU^* = U^*U = 1_E, \qquad\qquad U \in Un\,\check{H}_{\to k}\ .$$

Put

$$V_1 := (A_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q)W + (B_{k+1})_{\to}\sigma^F_{\to k}Q \in \check{G}_{k+1}\ .$$

Then

$$V_1^* = (A_{k+1})_{\to}W^*(1_E - \sigma^F_{\to k}Q) + (B_{k+1})_{\to}\sigma^F_{\to k}Q\,,$$

$$V_1V_1^* = (A_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q) + (B_{k+1})_{\to}\sigma^F_{\to k}Q \in Pr\,E_{k+1}\,,$$

$$V_1^*V_1 = (A_{k+1})_{\to}W^*(1_E - \sigma^F_{\to k}Q)W + (B_{k+1})_{\to}\sigma^F_{\to k}Q =$$

$$= (A_{k+1})_{\to}(1_E - W^*(\sigma^F_{\to k}Q)W) + (B_{k+1})_{\to}\sigma^F_{\to k}Q\ .$$

Put

$$Z := (1_E - \sigma^F_{\to k}Q) + ((C_{k+1})_{\to} + (C^*_{k+1})_{\to})\sigma^F_{\to k}Q \in E_{k+1}\ .$$

By Proposition 6.2.5 a),

$$Z^2 = (1_E - \sigma^F_{\to k}Q) + \sigma^F_{\to k}Q = 1_E, \qquad\qquad Z \in Un\,E_{k+1}\,,$$

$$ZV_1 = (A_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q)W + (C^*_{k+1})_{\to}\sigma^F_{\to k}Q\,,$$

$$V := ZV_1Z = (A_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q)W(1_E - \sigma^F_{\to k}Q) +$$

$$+(C^*_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q)W\sigma^F_{\to k}Q + (A_{k+1})_{\to}\sigma^F_{\to k}Q \in \check{G}_{\to k+1}\,,$$

$$\check{\psi}_{\to}V = (A_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q)\check{\psi}_{\to k}W + (A_{k+1})_{\to}\sigma^F_{\to k}Q = (A_{k+1})_{\to}U\,,$$

$$VV^* = ZV_1V_1^*Z \in Pr\,E_{k+1}, \qquad\qquad V^*V = ZV_1^*V_1Z\,,$$

$$1_E - VV^* = Z(1_E - V_1V_1^*)Z =$$

$$= Z((A_{k+1})_{\to}\sigma^F_{\to k}Q + (B_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q))Z\,,$$

$$1_E - V^*V = Z(1_E - V_1^*V_1)Z =$$

$$= Z((A_{k+1})_{\to}W^*(\sigma^F_{\to k}Q)W + (B_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q))Z =$$

$$= Z((A_{k+1})_{\to}\check{\varphi}_{\to k}Q + (B_{k+1})_{\to}(1_E - \sigma^F_{\to k}Q))Z\,,$$

$$\check{\varphi}_{\rightarrow,k+1}(Z((A_{k+1})_{\rightarrow}Q+(B_{k+1})_{\rightarrow}(1_E-\sigma^F_{\rightarrow k}Q))Z)=$$
$$=Z((A_{k+1})_{\rightarrow}\check{\varphi}_k Q+(B_{k+1})_{\rightarrow}(1_E-\sigma^F_{\rightarrow k}Q))Z=1_E-V^*V\,,$$
$$\check{\varphi}_{\rightarrow,k+1}(Z((A_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q+(B_{k+1})_{\rightarrow}(1_E-\sigma^F_{\rightarrow k}Q))Z))=1_E-VV^*\,.$$

By Proposition 7.2.6,

$$\delta_1[U]_1=[Z((A_{k+1})_{\rightarrow}Q+(B_{k+1})_{\rightarrow}(1_E-\sigma^F_{\rightarrow k}Q))Z]_0-$$
$$-[Z((A_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q+(B_{k+1})_{\rightarrow}(1_E-\sigma^F_{\rightarrow k})Q)Z]_0=[Q]_0-[\sigma^F_{\rightarrow}Q]_0=a\,.$$

Thus $a\in Im\,\delta_1$. ■

THEOREM 7.2.9 *The sequence*

$$K_1(F)\xrightarrow{K_1(\varphi)}K_1(G)\xrightarrow{K_1(\psi)}K_1(H)\xrightarrow{\delta_1}K_0(F)\xrightarrow{K_0(\varphi)}K_0(G)\xrightarrow{K_0(\psi)}K_0(H)$$

is exact.

The exactness was proved: for $K_1(G)$ in Proposition 7.1.9, for $K_1(H)$ in Proposition 7.2.7 and Proposition 7.2.5 a), for $K_0(F)$ in Proposition 7.2.8 and Proposition 7.2.5 b), and for $K_0(G)$ in Proposition 6.2.8 c). ■

7.3 $K_1(F) \approx K_0(SF)$

DEFINITION 7.3.1 *Let F be an E-C*-algebra. We denote by CF the E-C*-algebra of continuous maps $x:[0,1]\longrightarrow F$ with $x(0)=0$ and by SF its E-C*-subalgebra* $\{\,x\in CF\mid x(1)=0\}$ (Definition 2.1.1 or [2] Corollary 1.2.5 a),d)). *Moreover we denote by $\theta_F:K_1(F)\longrightarrow K_0(SF)$ the index map associated to the exact sequence*

$$0\longrightarrow SF\xrightarrow{i_F}CF\xrightarrow{j_F}F\longrightarrow 0\,,$$

in $\mathfrak{M}_E$, where i_F is the inclusion map and

$$j_F:CF\longrightarrow F\,,\quad x\longmapsto x(1)\,.$$

If $F\xrightarrow{\varphi}G$ is a morphism in $\mathfrak{M}_E$ then we put

$$S\varphi:SF\longrightarrow SG\,,\quad x\longmapsto\varphi\circ x\,,$$
$$C\varphi:CF\longrightarrow CG\,,\quad x\longmapsto\varphi\circ x\,.$$

If $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ are morphisms in $\mathfrak{M}_E$ then $S(\psi) \circ S(\varphi) = S(\psi \circ \varphi)$.

THEOREM 7.3.2 *θ_F is a group isomorphism for every E-C*-algebra F.*

CF is null-homotopic ([4] Example 4.1.5 or Proposition 2.4.1), so by the Homotopy invariance (Theorem 6.2.11 e), Proposition 7.1.8 e)), it is K-null. By Theorem 7.2.9, the sequence

$$K_1(CF) \xrightarrow{K_1(j_F)} K_1(F) \xrightarrow{\theta_F} K_0(SF) \xrightarrow{K_0(i_F)} K_0(CF)$$

is exact, so θ_F is a group isomorphism. ■

PROPOSITION 7.3.3 *Let F and G be E-C*-algebras.*

a) For all $(x,y) \in (SF) \times (SG)$ put

$$\overbrace{(x,y)} : [0,1] \longrightarrow F \times G, \quad s \longmapsto (x(s), y(s)) .$$

Then the map

$$(SF) \times (SG) \longrightarrow S(F \times G), \quad (x,y) \longmapsto \overbrace{(x,y)}$$

is an isomorphism in $\mathfrak{M}_E$ (Definition 1.1.2).

b) $K_1(F) \times K_1(G) \approx K_1(F \times G)$ **(Product Theorem)**.

a) is easy to see.

b) By Theorem 7.3.2, the maps

$$K_1(F) \times K_1(G) \xrightarrow{\theta_F \times \theta_G} K_0(SF) \times K_0(SG), \qquad K_1(F \times G) \xrightarrow{\theta_{F \times G}} K_0(S(F \times G))$$

are group isomorphisms. By a), $K_0((SF) \times (SG)) \approx K_0(S(F \times G))$ and by Corollary 6.2.10 b), $K_0((SF) \times (SG)) \approx K_0(SF) \times K_0(SG)$. Thus

$$K_1(F) \times K_1(G) \approx K_1(F \times G) .$$ ■

COROLLARY 7.3.4 *Let $F \xrightarrow{\varphi} F'$, $G \xrightarrow{\psi} G'$ be morphisms in $\mathfrak{M}_E$ and*

$$\varphi \times \psi : F \times G \longrightarrow F' \times G' , \quad (x,y) \longmapsto (\varphi x, \psi y) .$$

Then $\varphi \times \psi$ is a morphism in $\mathfrak{M}_E$ and

$$K_i(\varphi \times \psi) = K_i(\varphi) \times K_i(\psi)$$

for all $i \in \{0,1\}$.

The assertion follows easily from Corollary 6.2.10 b) and Proposition 7.3.3 b). ■

PROPOSITION 7.3.5 (Product Theorem) *Let $(F_j)_{j\in J}$ be a finite family of E-C*-algebras, $F := \prod_{j\in J} F_j$* (Definition 1.1.2)*, and for every $j \in J$ let $\varphi_j : F_j \longrightarrow F$ be the canonical inclusion and $\psi_j : F \longrightarrow F_j$ the projection. Then for every $i \in \{0,1\}$,*

$$\Phi : \prod_{j\in J} K_i(F_j) \longrightarrow K_i(F) , \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} K_i(\varphi_j) a_j$$

is a group isomorphism and

$$\Psi : K_i(F) \longrightarrow \prod_{j\in J} K_i(F_j) , \quad a \longmapsto (K_i(\psi_j)a)_{j\in J}$$

is its inverse.

Φ and Ψ are obviously group homomorphisms. For $j,k \in J$, $\psi_j \circ \varphi_k = 0$ if $j \neq k$ and $\psi_j \circ \varphi_j = id_{F_j}$. Thus for $(a_j)_{j\in J} \in \prod_{j\in J} K_i(F_j)$ and $k \in J$,

$$(\Psi\Phi(a_j)_{j\in J})_k = K_i(\psi_k) \sum_{j\in J} K_i(\varphi_j) a_j = a_k$$

i.e. $\Psi \circ \Phi$ is the identity map of $\prod_{j\in J} K_i(F_j)$. Since $\sum_{j\in J} \varphi_j \circ \psi_j = id_F$, for $a \in K_i(F)$,

$$\Phi\Psi a = \Phi(K_i(\psi_j)a)_{j\in J} = \sum_{j\in J} K_i(\varphi_j) K_i(\psi_j) a = K_i\left(\sum_{j\in J} \varphi_j \circ \psi_j\right) a = a$$

i.e. $\Phi \circ \Psi = id_{K_i(F)}$. ■

THEOREM 7.3.6 (Continuity of K_1) *Let $\{(F_i)_{i\in I}, (\varphi_{ij})_{i,j\in I}\}$ be an inductive system in $\mathfrak{M}_E$ and let $\{F, (\varphi_i)_{i\in I}\}$ be its limit in $\mathfrak{M}_E$. By* Proposition 7.1.8 a),

$$\{(K_1(F_i))_{i\in I}, (K_1(\varphi_{ij}))_{i,j\in I}\}$$

is an inductive system in the category of additive groups. Let $\{\mathscr{G}, (\psi_i)_{i\in I}\}$ be its limit in this category and let $\psi : \mathscr{G} \longrightarrow K_1(F)$ be the group homomorphism such that $\psi \circ \psi_i = K_1(\varphi_i)$ for every $i \in I$. Then ψ is a group isomorphism.

By [4] Exercise 10.2, $\{SF, (S\varphi_i)_{i\in I}\}$ is the limit in $\mathfrak{M}_E$ of the inductive system $\{(SF_i)_{i\in I}, (S\varphi_{ij})_{i,j\in I}\}$. By Theorem 6.2.12, $\{K_0(SF), (K_0(S\varphi_i))_{i\in I}\}$ may be identified with the inductive limit in the category of additive groups of the inductive system $\{K_0(SF_i)_{i\in I}, (K_0(S\varphi_{ij}))_{i,j\in I}\}$ and the assertion follows from Theorem 7.3.2. ■

PROPOSITION 7.3.7 *Let F be an E-C*-algebra , $n \in \mathbb{N}$, $U \in Un\,\check{F}_{n-1}$, $V \in Un\,(\overbrace{CF}^{\vee})_n$, and $P \in Pr\,(\overbrace{SF}^{\vee})_n$ such that*

$$\check{j}_F V = A_n U + B_n U^*, \qquad \check{i}_F P = V A_n V^* .$$

Then

$$\theta_F [U]_1 = [P]_0 - [\sigma_n^{SF} P]_0.$$

The assertion follows from Corollary 7.2.3 and Definition 7.3.1. ■

PROPOSITION 7.3.8 *If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ then the diagram*

$$\begin{array}{ccc} K_1(F) & \xrightarrow{K_1(\varphi)} & K_1(G) \\ {\scriptstyle\theta_F}\downarrow & & \downarrow{\scriptstyle\theta_G} \\ K_0(SF) & \xrightarrow[K_0(S\varphi)]{} & K_0(SG) \end{array}$$

is commutative.

The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SF & \xrightarrow{i_F} & CF & \xrightarrow{j_F} & F & \longrightarrow & 0 \\ & & {\scriptstyle S\varphi}\downarrow & & {\scriptstyle C\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & SG & \xrightarrow[i_G]{} & CG & \xrightarrow[j_G]{} & G & \longrightarrow & 0 \end{array}$$

is commutative and the assertion follows from Proposition 7.2.4. ■

Remark. By Theorem 7.3.2 and Proposition 7.3.8, the functor K_1 is determined by the functor K_0.

COROLLARY 7.3.9 (Split Exact Theorem) *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\gamma}{\overset{\psi}{\rightleftarrows}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ *then*

$$0 \longrightarrow K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \underset{K_1(\gamma)}{\overset{K_1(\psi)}{\rightleftarrows}} K_1(H) \longrightarrow 0$$

is also split exact. In particular the map

$$K_1(F) \times K_1(H) \longrightarrow K_1(G), \quad (a,b) \longmapsto K_1(\varphi)a + K_1(\lambda)b$$

is a group isomorphism and $K_1\left(\check{F}\right) \approx K_1(E) \times K_1(F)$.

By Theorem 7.2.9, the sequence

$$K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \xrightarrow{K_1(\psi)} K_1(H) \xrightarrow{\delta_1} K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \xrightarrow{K_0(\psi)} K_0(H)$$

is exact and by Proposition 7.1.8 a) and Proposition 7.1.6 d),

$$K_1(\psi) \circ K_1(\gamma) = K_1(\psi \circ \gamma) = K_1(id_H) = id_{K_1(H)}\,.$$

It remains only to prove that $K_1(\varphi)$ is injective.

It is easy to see that

$$0 \longrightarrow SF \xrightarrow{S\varphi} SG \underset{S\gamma}{\overset{S\psi}{\rightleftarrows}} SH \longrightarrow 0$$

is split exact. By Proposition 6.2.9, $K_0(S\varphi)$ is injective and by Proposition 7.3.8, the diagram

$$\begin{array}{ccc} K_1(F) & \xrightarrow{K_1(\varphi)} & K_1(G) \\ {\scriptstyle\theta_F}\downarrow & & \downarrow{\scriptstyle\theta_G} \\ K_0(SF) & \xrightarrow[K_0(S\varphi)]{} & K_0(SG) \end{array}$$

is commutative. Since θ_F is injective (Theorem 7.3.2), $K_1(\varphi)$ is also injective.

The last assertion follows from the fact that

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\overset{\longleftarrow}{\lambda^F}}{\overset{\pi^F}{\longrightarrow}} E \longrightarrow 0$$

is split exact. ∎

COROLLARY 7.3.10 *Let*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\overset{\longleftarrow}{\gamma}}{\overset{\psi}{\longrightarrow}} H \longrightarrow 0\,, \qquad 0 \longrightarrow F' \xrightarrow{\varphi'} G' \underset{\overset{\longleftarrow}{\gamma'}}{\overset{\psi'}{\longrightarrow}} H' \longrightarrow 0$$

be split exact sequences in $\mathfrak{M}_E$ *and*

$$F \xrightarrow{\lambda} F', \qquad G \xrightarrow{\mu} G', \qquad H \xrightarrow{\nu} H'$$

morphisms in $\mathfrak{M}_E$ *such that the corresponding diagram is commutative and let* $i \in \{0,1\}$.

a) *If we denote by*

$$\phi : K_i(F) \times K_i(H) \longrightarrow K_i(G)\,, \quad (a,b) \longmapsto K_i(\varphi)a + K_i(\gamma)b\,,$$

$$\phi' : K_i(F') \times K_i(H') \longrightarrow K_i(G')\,, \quad (a',b') \longmapsto K_i(\varphi')a' + K_i(\gamma')b'$$

the group isomorphisms (Proposition 6.2.9, Corollary 7.3.9) *then*

$$K_i(\mu) \circ K_i(\phi) = K_i(\phi') \circ (K_i(\lambda) \times K_i(\nu))\;.$$

b) *If we identify* $K_i(G)$ *with* $K_i(F) \times K_i(H)$ *using* ϕ *and* $K_i(G')$ *with* $K_i(F') \times K_i(H')$ *using* ϕ' *then*

$$K_i(\mu) : K_i(G) \longrightarrow K_i(G')\,, \quad (a,b) \longmapsto (K_i(\lambda)a, K_i(\nu)b)\;.$$

a) For $(a,b) \in K_i(F) \times K_i(H)$,

$$K_i(\mu)K_i(\phi)(a,b) = K_i(\mu)(K_i(\varphi)a + K_i(\gamma)b) =$$

$$= K_i(\varphi')K_i(\lambda)a + K_i(\gamma')K_i(\nu)b = K_i(\phi')(K_i(\lambda) \times K_i(\nu))(a,b)\;.$$

b) follows from a). ∎

Chapter 8

Bott Periodicity

8.1 The Bott Map

LEMMA 8.1.1 *Let F be a full E-C*-algebra and $n \in \mathbb{N}$. We identify SF with $\mathscr{C}_0(\mathbb{T} \setminus \{1\}, F)$ in an obvious way.*

a) *$F_{\mathbb{T}} := \{\, X \in \mathscr{C}(\mathbb{T}, F) \mid X(1) \in E \,\}$ is a full E-C*-subalgebra of $\mathscr{C}(\mathbb{T}, F)$.*

b) *If we put for every $(\alpha, x) \in \overbrace{SF}^{\vee}$*

$$\overbrace{(\alpha, x)} : \mathbb{T} \longrightarrow F\,, \quad z \longmapsto \alpha + x(z)$$

then the map

$$\psi : \overbrace{SF}^{\vee} \longrightarrow F_{\mathbb{T}}\,, \quad (\alpha, x) \longmapsto \overbrace{(\alpha, x)}$$

is an E-C-isomorphism. Thus the map*

$$\psi_n : \left(\overbrace{SF}^{\vee}\right)_n \longrightarrow (F_{\mathbb{T}})_n$$

is also an E-C-isomorphism.*

c) *For every $Y \in (F_{\mathbb{T}})_n$ put*

$$\ddot{Y} : \mathbb{T} \longrightarrow F_n\,, \quad z \longmapsto \sum_{t \in T_n} (Y_t(z) \otimes id_K) V_t\ .$$

Then $\ddot{Y} \in \{\, X \in \mathscr{C}(\mathbb{T}, F_n) \mid X(1) \in E_n \,\}$ for every $Y \in (F_{\mathbb{T}})_n$ and the map

$$\phi^n : (F_{\mathbb{T}})_n \longrightarrow \{\, X \in \mathscr{C}(\mathbb{T}, F_n) \mid X(1) \in E_n \,\}\,, \quad Y \longmapsto \ddot{Y}$$

is an E-C-isomorphism.*

d) *The map*

$$\phi^n \circ \psi_n : \left(\overbrace{SF}^{\vee}\right)_n \longrightarrow \{\, X \in \mathscr{C}(\mathbb{T}, F_n) \mid X(1) \in E_n \,\}$$

is an E-C-isomorphism. We identify these two full E-C*-algebras by using this isomorphism.The map*

$$Un\, \left(\overbrace{SF}^{\vee}\right)_n \longrightarrow \{\, X \in \mathscr{C}(\mathbb{T}, Un\, F_n) \mid X(1) \in Un\, E_n \,\}$$

defined by $\phi^n \circ \psi_n$ is a homeomorphism.

e) For every

$$X := \sum_{t\in T_n} ((\alpha_t, X_t)\otimes id_K)V_t \in \left(\overbrace{SF}^{\vee}\right)_n$$

and $z\in\mathbb{T}$*,*

$$(\phi^n \psi_n X)(z) = \sum_{t\in T_n} ((\alpha_t + X_t(z))\otimes id_K)V_t \in F_n\,,$$

$$(\phi^n \psi_n X)(1) = \sum_{t\in T_n} (\alpha_t \otimes id_K)V_t \in E_n\,.$$

f) Consider the split exact sequence in $\mathfrak{M}_E$ (Definition 4.1.4)

$$0 \longrightarrow SF \xrightarrow{\iota^{SF}} \overbrace{SF}^{\vee} \underset{\lambda^{SF}}{\overset{\pi^{SF}}{\rightleftarrows}} E \longrightarrow 0\,.$$

Then

$$\left(\pi^{SF}\right)_n X = (\phi^n \psi_n X)\,(1)$$

for every $X \in \left(\overbrace{SF}^{\vee}\right)_n$*.*

g) If $F \overset{\varphi}{\to} G$ *is a morphism in* $\mathfrak{C}_E$ *then, by the identification of d), for every* $X\in\mathscr{C}(\mathbb{T},F_n)$ *with* $X(1)\in E_n$ *and for every* $z\in\mathbb{T}$*,*

$$\left(\left(\overbrace{S\varphi}^{\vee}\right)_n X\right)(z) = \varphi_n X(z)\,.$$

a) is obvious.

b) For $(\alpha,x),(\beta,y)\in \overbrace{SF}^{\vee}$, $\gamma\in E$, and $z\in\mathbb{T}$,

$$(\overbrace{(\alpha,x)})^*(z) = \alpha^* + x(z)^* = \overbrace{(\alpha,x)^*}(z)\,,$$

$$(\overbrace{(\alpha,x)}(z))(\overbrace{(\beta,y)}(z)) = (\alpha+x(z))(\beta+y(z)) = \alpha\beta+\alpha y(z)+x(z)\beta+x(z)y(z) =$$

$$= \overbrace{(\alpha\beta,\alpha y+\beta x+xy)}(z) = \overbrace{(\alpha,x)(\beta,y)}(z)\,,$$

$$\overbrace{(\gamma,0)}(z) = \gamma,$$

so ψ is an E-C*-homomorphism. If $\overbrace{(\alpha,x)} = 0$ then for all $z\in\mathbb{T}$

$$\alpha = \alpha + x(1) = 0\,, \qquad x(z) = \alpha + x(z) = 0\,, \qquad x = 0\,,$$

so ψ is injective.

Let $X \in F_{\mathbb{T}}$ and put $\alpha := X(1) \in E$ and

$$x : \mathbb{T} \longrightarrow F, \quad z \longmapsto X(z) - X(1) .$$

Then $(\alpha, x) \in \overbrace{\check{SF}}$ and for $z \in \mathbb{T}$,

$$\overbrace{(\alpha, x)}(z) = \alpha + x(z) = X(1) + X(z) - X(1) = X(z) .$$

Thus $\overbrace{(\alpha, x)} = X$ and ψ is surjective.

By [2] Corollary 2.2.5 and [2] Theorem 2.1.9 a), ψ_n is an isomorphism.

c) follows from [2] Proposition 2.3.7 and [2] Theorem 2.1.9 a).

d) follows from b) and c).

e) We have

$$\psi_n X = \sum_{t \in T_n} (\overbrace{(\alpha_t, X_t)} \otimes id_K) V_t ,$$

$$(\phi^n \psi_n X)(z) = \sum_{t \in T_n} ((\alpha_t + X_t(z)) \otimes id_K) V_t \in F_n ,$$

$$(\phi^n \psi_n X)(1) = \sum_{t \in T_n} (\alpha_t \otimes id_K) V_t \in E_n .$$

f) and g) follow from e). ■

DEFINITION 8.1.2 *We put for every full E-C*-algebra F, $n \in \mathbb{N}$, and $P \in F_n$,*

$$\widetilde{P} : \mathbb{T} \longrightarrow F_n , \quad z \longmapsto zP + (1_E - P) .$$

By the identification of Lemma 8.1.1 d),

$$\widetilde{P} \in \{\, X \in \mathscr{C}(\mathbb{T}, Un\, F_n) \mid X(1) \in E_n \,\} = Un \left(\overbrace{\check{SF}} \right)_n$$

for every $P \in Pr\, F_n$. Obviously, $\widetilde{0} = 1_E$ and $\widetilde{1_E} = z1_E$.

PROPOSITION 8.1.3 *If F is a full E-C*-algebra, $n \in \mathbb{N}$, and $P \in Pr\, F_{n-1}$ then*

$$\bar{\tau}_n^{\overbrace{SF}^{\vee}} \widetilde{P} = \widetilde{\bar{\rho}_n^F P},$$

(with the identification of Lemma 8.1.1 d)*). Thus we get a well-defined map*

$$\nu_F : Pr\, F_{\rightarrow} \longrightarrow un\, \overbrace{SF}^{\vee}$$

with $\nu_F P = \widetilde{P}$ for every $P \in Pr\, F_{\rightarrow} = \bigcup_{n \in \mathbb{N}} Pr\, F_{\rightarrow n}$.

For $z \in \mathbb{T}$,

$$(\bar{\tau}_n^{\overbrace{SF}^{\vee}} \widetilde{P})(z) = (A_n \widetilde{P} + B_n)(z) = A_n(zP + (1_E - P)) + B_n =$$

$$= zA_n P + (1_E - A_n P) = \widetilde{\bar{\rho}_n^F P}(z)\,. \qquad \blacksquare$$

PROPOSITION 8.1.4 *For every full E-C*-algebra F there is a unique group homomorphism*

$$\beta_F : K_0(F) \longrightarrow K_1(SF) \qquad \textbf{(the Bott map)}$$

such that for every $P \in Pr\, F_{\rightarrow}$,

$$\beta_F [P]_0 = (\nu_F P) / \sim_1 = \left[\tilde{P}\right]_1 \,.$$

Let $P, Q \in Pr\, F_{\rightarrow}$ with $P \sim_0 Q$. By Proposition 6.2.6, there are $m, n \in \mathbb{N}$, $m \geq n+2$, and $U \in Un_0\, F_m$ with $P, Q \in Pr\, F_n$ and $UPU^* = Q$ and so

$$(U\widetilde{P}U^*)(z) = U\widetilde{P}(z)U^* = zUPU^* + (1_E - UPU^*) = \widetilde{Q}(z)$$

for every $z \in \mathbb{T}$. Thus $U\widetilde{P}U^* = \widetilde{Q}$, $\widetilde{P} \sim_h \widetilde{Q}$, and $\widetilde{P} \sim_1 \widetilde{Q}$.

Let $P, Q \in Pr\, F_{\rightarrow}$ with $PQ = 0$. We may assume $P, Q \in Pr\, F_{n-1}$ with $P = PA_n$ and $Q = QB_n$ for some $n \in \mathbb{N}$ (Proposition 6.1.3). For every $z \in \mathbb{T}$,

$$\widetilde{P}(z) = zPA_n + (1_E - PA_n), \qquad \widetilde{Q}(z) = zQB_n + (1_E - QB_n)\,,$$

$$(\widetilde{P}\widetilde{Q})(z) = \widetilde{P}(z)\widetilde{Q}(z) = zPA_n + zQB_n + 1_E - QB_n - PA_n =$$

$$= z(P+Q) + (1_E - (P+Q)) = (\widetilde{P+Q})(z), \qquad \widetilde{P}\widetilde{Q} = \widetilde{P+Q}\,.$$

By Proposition 6.1.9, there is a unique group homomorphism

$$\beta_F : K_0(F) \longrightarrow K_1(SF)$$

with the required property. ■

PROPOSITION 8.1.5 *Let F be an E-C*-algebra.*

a) *There is a unique map $\beta_F : K_0(F) \longrightarrow K_1(SF)$ (called* **the Bott map***) such that the diagram*

$$\begin{array}{ccc} K_0(F) & \xrightarrow{K_0(\iota^F)} & K_0(\check{F}) \\ {\scriptstyle \beta_F}\big\downarrow & & \big\downarrow{\scriptstyle \beta_{\check{F}}} \\ K_1(SF) & \xrightarrow[K_1(S\iota^F)]{} & K_1(S\check{F}) \end{array}$$

is commutative. β_F is a group homomorphism.

b) *If F is a full E-C*-algebra then the above map β_F coincides with the map β_F defined in* Proposition 8.1.4.

c) *If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ then the diagram*

$$\begin{array}{ccc} K_0(F) & \xrightarrow{K_0(\varphi)} & K_0(G) \\ {\scriptstyle \beta_F}\big\downarrow & & \big\downarrow{\scriptstyle \beta_G} \\ K_1(SF) & \xrightarrow[K_1(S\varphi)]{} & K_1(SG) \end{array}$$

is commutative.

c) for $\mathfrak{C}_E$ with $F \xrightarrow{\varphi} G$ unital. For $n \in \mathbb{N}$, $P \in Pr\, F_n$, and $z \in \mathbb{T}$, by Lemma 8.1.1 g),

$$\left(\left(\overset{\vee}{\overbrace{S\varphi}}\right)_n \widetilde{P}\right)(z) = z\varphi_n P + (1_E - \varphi_n P) = \left(\widetilde{\varphi_n P}\right)(z)\,,$$

$$\left(\overset{\vee}{\overbrace{S\varphi}}\right)_n \widetilde{P} = \widetilde{\varphi_n P}\,.$$

By Proposition 6.1.10 c), Proposition 8.1.4, and Proposition 7.1.6 c),

$$K_1(S\varphi)\beta_F[P]_0 = K_1(S\varphi)\left[\widetilde{P}\right]_1 =$$

$$= \left[\left(\overbrace{S\varphi}^{\vee}\right)_n\widetilde{P}\right]_1 = \left[\widetilde{\varphi_n P}\right]_1 = \beta_G[\varphi_n P]_0 = \beta_G K_0(\varphi)[P]_0\,,$$

$$K_1(S\varphi)\circ\beta_F = \beta_G\circ K_0(\varphi)\,.$$

a) By c) for $\mathfrak{C}_E$, the diagram

$$\begin{array}{ccc} K_0(\check{F}) & \xrightarrow{K_0(\pi^F)} & K_0(E) \\ {\scriptstyle\beta_{\check{F}}}\downarrow & & \downarrow{\scriptstyle\beta_E} \\ K_1(S\check{F}) & \xrightarrow[K_1(S\pi^F)]{} & K_1(SE) \end{array}$$

is commutative. By Proposition 6.1.12 c) and Corollary 7.3.9 the sequences

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\iota^F)} K_0(\check{F}) \xrightarrow{K_0(\pi^F)} K_0(E) \longrightarrow 0\,,$$

$$0 \longrightarrow K_1(SF) \xrightarrow{K_1(S\iota^F)} K_1(S\check{F}) \xrightarrow{K_1(S\pi^F)} K_1(SE) \longrightarrow 0$$

are exact, since the sequence

$$0 \longrightarrow SF \xrightarrow{S\iota^F} S\check{F} \underset{\xleftarrow[S\lambda^F]{}}{\xrightarrow{S\pi^F}} SE \longrightarrow 0$$

is split exact. By the above c) for $\mathfrak{C}_E$, Corollary 6.2.3 a), and Proposition 6.2.2 e),

$$K_1(S\pi^F)\circ\beta_{\check{F}}\circ K_0(\iota^F) = \beta_E\circ K_0(\pi^F)\circ K_0(\iota^F) =$$

$$= \beta_E\circ K_0(\pi^F\circ\iota^F) = \beta_E\circ K_0(0) = 0\,.$$

Thus

$$Im\,(\beta_{\check{F}}\circ K_0(\iota^F)) \subset Ker\,K_1(S\pi^F) = Im\,K_1(S\iota^F)\,.$$

The assertion follows now from the fact that $K_1(S\iota^F)$ is injective.

b) By c) for $\mathfrak{C}_E$, the diagram

$$\begin{array}{ccc} K_0(F) & \xrightarrow{K_0(\iota^F)} & K_0(\check{F}) \\ {\scriptstyle\beta_F}\downarrow & & \downarrow{\scriptstyle\beta_{\check{F}}} \\ K_1(SF) & \xrightarrow[K_1(S\iota^F)]{} & K_1(S\check{F}) \end{array}$$

is commutative, with β_F defined in Proposition 8.1.4. By a), this β_F coincides with β_F defined in a).

c) The following diagrams

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ {\scriptstyle \iota^F}\downarrow & & \downarrow{\scriptstyle \iota^G} \\ \check{F} & \xrightarrow[\check{\varphi}]{} & \check{G} \end{array} \qquad \begin{array}{ccc} SF & \xrightarrow{S\varphi} & SG \\ {\scriptstyle S\iota^F}\downarrow & & \downarrow{\scriptstyle S\iota^G} \\ S\check{F} & \xrightarrow[S\check{\varphi}]{} & S\check{G} \end{array} \qquad \begin{array}{ccc} K_1(SF) & \xrightarrow{K_1(S\varphi)} & K_1(SG) \\ {\scriptstyle K_1(S\iota^F)}\downarrow & & \downarrow{\scriptstyle K_1(S\iota^G)} \\ K_1(S\check{F}) & \xrightarrow[K_1(S\check{\varphi})]{} & K_1(S\hat{G}) \end{array}$$

are obviously commutative (Proposition 7.1.8 a)). So by a) and c) for $\mathfrak{C}_E$ (and Corollary 6.2.3 a), Proposition 7.1.8 a)),

$$K_1(S\iota^G)\circ\beta_G\circ K_0(\varphi) = \beta_{\check{G}}\circ K_0(\iota^G)\circ K_0(\varphi) = \beta_{\check{G}}\circ K_0(\check{\varphi})\circ K_0(\iota^F) =$$

$$= K_1(S\check{\varphi})\circ\beta_{\check{F}}\circ K_0(\iota^F) = K_1(S\check{\varphi})\circ K_1(S\iota^F)\circ\beta_F = K_1(S\iota^G)\circ K_1(S\varphi)\circ\beta_F\ .$$

The assertion follows now from the fact that $K_1(S\iota^G)$ is injective. ■

8.2 Higman's Linearization Trick

Throughout this section F denotes a full E-C*-algebra, $m,n\in\mathbb{N}$, and $l := 2^m - 1$.

DEFINITION 8.2.1 *We shall use the following notation* ([4] 11.2)*:*

$$Trig(n) := \left\{ X\in\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n)) \mid X(z) = \sum_{p=-m}^{m} a_p z^p,\, a_p\in F_n \right\},$$

$$Pol(n,m) := \left\{ X\in\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n)) \mid X(z) = \sum_{p=0}^{m} a_p z^p,\, a_p\in F_n \right\},$$

$$Pol(n) := \bigcup_{m\in\mathbb{N}} Pol(n,m), \qquad Lin(n) := Pol(n,1)\,,$$

$$Proj(n) := \left\{ \widetilde{P} \,\middle|\, P\in Pr\, F_n \right\}.$$

LEMMA 8.2.2

a) If $X \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$ then there are $k \in \mathbb{N}$ and $Y \in Pol(n)$ such that $z^k X$ is homotopic to Y in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$.

b) If $P, Q \in Pr\, F_n$ such that $\widetilde{P}$ and $\widetilde{Q}$ are homotopic in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$ then there are $k, m \in \mathbb{N}$ such that $z^k \widetilde{P}$ is homotopic to $z^k \widetilde{Q}$ in $Pol(n, l)$.

a) It is possible to adapt [4] Lemma 11.2.3 to the present situation in order to find a $Z \in Trig(n)$ such that

$$\|X - Z\| < \left\|X^{-1}\right\|^{-1} .$$

By [4] Proposition 2.1.11, X and Z are homotopic in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$. There is a $k \in \mathbb{N}$ such that $Y := z^k Z \in Pol(n)$. Then $z^k X$ and Y are homotopic in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$.

b) The proof of [4] Lemma 11.2.4 (ii) works in this case too. ■

DEFINITION 8.2.3 *The map*

$$\{0,1\}^m \longrightarrow \mathbb{N}_l \cup \{0\}, \quad j \longmapsto \sum_{i=1}^{m} j_i 2^{i-1}$$

is bijective. We denote by

$$\mathbb{N}_l \cup \{0\} \longrightarrow \{0,1\}^m, \quad p \longmapsto |p|$$

its inverse. For every $i \in \mathbb{N}_m$ and $p, q \in \mathbb{N}_l \cup \{0\}$ we put

$$(p,q)_i := \begin{cases} A_{n+i} & \text{if} \quad |p|_i = |q|_i = 0 \\ C^*_{n+i} & \text{if} \quad |p|_i = 0,\, |q|_i = 1 \\ C_{n+i} & \text{if} \quad |p|_i = 1,\, |q|_i = 0 \\ B_{n+i} & \text{if} \quad |p|_i = |q|_i = 1 \end{cases} .$$

LEMMA 8.2.4

a) For $p, q, r, s \in \mathbb{N}_l \cup \{0\}$ and $i \in \mathbb{N}_m$,

$$(p,q)_i (r,s)_i = \begin{cases} 0 & \text{if} \quad |q|_i \neq |r|_i \\ (p,s)_i & \text{if} \quad |q|_i = |r|_i \end{cases} .$$

In particular

$$\prod_{i=1}^{m}((p,q)_i(r,s)_i)=\begin{cases}0 & \text{if} \quad q\neq r\\ \prod\limits_{i=1}^{m}(p,s)_i & \text{if} \quad q=r\end{cases}.$$

b) For $p,q\in\mathbb{N}_l\cup\{0\}$ and $i\in\mathbb{N}_m$,

$$A_{n+i}(p,q)_i=\begin{cases}(p,q)_i & \text{if} \quad |p|_i=0\\ 0 & \text{if} \quad |p|_i=1\end{cases},$$

$$(p,q)_iA_{n+i}=\begin{cases}(p,q)_i & \text{if} \quad |q|_i=0\\ 0 & \text{if} \quad |q|_i=1\end{cases}.$$

In particular

$$p\neq 0\Longrightarrow\prod_{i=1}^{m}(A_{n+i}(p,q)_i)=0\,,$$

$$q\neq 0\Longrightarrow\prod_{i=1}^{m}((p,q)_iA_{n+i})=0\,,$$

$$\sum_{r=q}^{l}\prod_{i=1}^{m}(A_{n+i}(r,r-q)_i)=\begin{cases}0 & \text{if} \quad q\neq 0\\ \prod\limits_{i=1}^{m}A_{n+i} & \text{if} \quad q=0\end{cases}.$$

c) $\sum\limits_{p=0}^{l}\prod\limits_{i=1}^{m}(p,p)_i=1_E.$

a) and b) is a long verification.

c) For every $p\in\mathbb{N}_l\cup\{0\}$ put

$$J_p:=\{\,i\in\mathbb{N}_m\mid |p|_i=0\,\}\,,\qquad K_p:=\{\,i\in\mathbb{N}_m\mid |p|_i=1\,\}\;.$$

Then

$$1_E=\prod_{i=1}^{m}(A_{n+i}+B_{n+i})=\sum_{p=0}^{l}\left(\prod_{i\in J_p}A_{n+i}\right)\left(\prod_{i\in K_p}B_{n+i}\right)=\sum_{p=0}^{l}\prod_{i=1}^{m}(p,p)_i\,.$$ ■

LEMMA 8.2.5 *Let $a\in(F_n)^l$ and*

$$X:=\sum_{p=1}^{l}a_p\sum_{q=p}^{l}\prod_{i=1}^{m}(q,q-p)_i\qquad (X\in F_{m+n})\,.$$

a) $X^{2^m} = 0.$

b) $1_E - X$ *is invertible.*

a) We put $D := \mathbb{N}_l$ and for every $k \in \mathbb{N}$ and $p \in D^k$,

$$p^{(k)} := \sum_{j=1}^{k} p_j, \qquad a_p^{(k)} := \prod_{j=1}^{k} a_{p_j} .$$

We want to prove by induction that for every $k \in \mathbb{N}$,

$$X^k = \sum_{p \in D^k} a_p^{(k)} \sum_{q=p^{(k)}}^{l} \prod_{i=1}^{m} (q, q - p^{(k)})_i .$$

The assertion holds for $k = 1$. Assume the assertion holds for $k \in \mathbb{N}$. Then

$$X^{k+1} = \sum_{p \in D^k} \sum_{p' \in D} a_p^{(k)} a_{p'} \sum_{q=p^{(k)}}^{l} \sum_{q'=p'}^{l} \prod_{i=1}^{m} ((q, q - p^{(k)})_i (q', q' - p')_i) .$$

By Lemma 8.2.4 a),

$$X^{k+1} = \sum_{p \in D^k} \sum_{p' \in D} a_p^{(k)} a_{p'} \sum_{q=p^{(k)}+p'}^{l} \prod_{i=1}^{m} (q, q - p^{(k)} - p')_i =$$

$$= \sum_{p \in D^{k+1}} a_p^{(k+1)} \sum_{q=p^{(k+1)}}^{l} \prod_{i=1}^{m} (q, q - p^{(k+1)})_i ,$$

which finishes the inductive proof. Since $p^{(k)} \geq k$ for every $k \in \mathbb{N}$ we get $X^{2^m} = 0$.

b) By a), $1_E + \sum_{k=1}^{l} X^k$ is the inverse of $1_E - X$. ■

PROPOSITION 8.2.6 (Higman's linearization trick) *There is a continuous map*

$$\mu : Pol(n, l) \longrightarrow Lin(n + m)$$

such that μX *is homotopic to* $X \left(\prod_{i=1}^{m} A_{n+i} \right) + \left(1_E - \prod_{i=1}^{m} A_{n+i} \right)$ *in* $Pol(n + m, 2l + 1)$ *for every* $X \in Pol(n, l)$*. If* $X \in Proj(n)$ *then the above homotopy takes place in* $Lin(n + 1)$*.*

Assume $X \in Pol(n,l)$ is given by

$$X = \sum_{p=0}^{l} a_p z^p \,,$$

where $a_p \in F_n$ for every $p \in \mathbb{N}_l \cup \{0\}$. Put

$$X_p := \sum_{q=p}^{l} a_q z^{q-p} \qquad\qquad (\in \mathscr{C}(\mathbb{T}, F_n))$$

for all $p \in \mathbb{N}_l \cup \{0\}$ and for all $s \in [0,1]$,

$$Y_s := 1_E - s \sum_{p=1}^{l} X_p \prod_{i=1}^{m} (0,p)_i \qquad\qquad (\in \mathscr{C}(\mathbb{T}, F_{n+m}))\,,$$

$$Z_s := 1_E + s \sum_{q=1}^{l} z^q \sum_{r=q}^{l} \prod_{i=1}^{m} (r, r-q)_i \qquad\qquad (\in \mathscr{C}(\mathbb{T}, F_{n+m}))\,.$$

By Lemma 8.2.4 a),

$$Y_s(1_E + s \sum_{p=1}^{l} X_p \prod_{i=1}^{m} (0,p)_i) = (1_E + s \sum_{p=1}^{l} X_p \prod_{i=1}^{m} (0,p)_i) Y_s =$$

$$= 1_E + s^2 \sum_{p,q=1}^{l} X_p X_q \prod_{i=1}^{m} ((0,p)_i (0,q)_i) = 1_E \,,$$

so Y_s is invertible. By Lemma 8.2.5 b), Z_s is also invertible. Thus for every $s \in [0,1]$, Y_s and Z_s are homotopic to 1_E in $\mathscr{C}(\mathbb{T}, GL(F_{n+m}))$ and belong therefore to $Pol(n+m,l)$. By Lemma 8.2.4 c),

$$Z_1 = \sum_{q=0}^{l} z^q \sum_{r=q}^{l} \prod_{i=1}^{m} (r, r-q)_i \,.$$

Put

$$\mu X := 1_E - \prod_{i=1}^{m} A_{n+i} + \sum_{p=0}^{l} a_p \prod_{i=1}^{m} (0,p)_i - z \sum_{p=1}^{l} \prod_{i=1}^{m} (p, p-1)_i \;\; (\in \mathscr{C}(\mathbb{T}, F_{n+m}))\,.$$

For $z \in \mathbb{T}$,

$$((\mu X) Z_1)(z) = \sum_{p=0}^{l} z^p \sum_{q=p}^{l} \prod_{i=1}^{m} (q, q-p)_i - \sum_{p=0}^{l} z^p \sum_{q=p}^{l} \prod_{i=1}^{m} (A_{n+i}(q, q-p)_i) +$$

$$+\sum_{p,q=0}^{l} a_p z^q \sum_{r=q}^{l}\prod_{i=1}^{m}((0,p)_i(r,r-q)_i) - \sum_{q=0}^{l} z^{q+1}\sum_{p=1}^{l}\sum_{r=q}^{l}\prod_{i=1}^{m}((p,p-1)_i(r,r-q)_i)\,.$$

By Lemma 8.2.4 b),

$$\sum_{p=0}^{l} z^p \sum_{q=p}^{l}\prod_{i=1}^{m}(A_{n+i}(q,q-p)_i) = \prod_{i=1}^{m} A_{n+i}$$

and by Lemma 8.2.4 a),

$$\sum_{p,q=0}^{l} a_p z^q \sum_{r=q}^{l}\prod_{i=1}^{m}((0,p)_i(r,r-q)_i) = \sum_{q=0}^{l} z^q \sum_{p=q}^{l} a_p \prod_{i=1}^{m}(0,p-q)_i =$$

$$= \sum_{q=0}^{l} z^q \sum_{r=0}^{l-q} a_{q+r}\prod_{i=1}^{m}(0,r)_i = \sum_{r=0}^{l}\sum_{q=0}^{l-r} z^q a_{q+r}\prod_{i=1}^{m}(0,r)_i =$$

$$= \sum_{r=0}^{l}\sum_{s=r}^{l} z^{s-r} a_s \prod_{i=1}^{m}(0,r)_i = \sum_{r=0}^{l} X_r \prod_{i=1}^{m}(0,r)_i\,,$$

$$\sum_{q=0}^{l} z^{q+1}\sum_{p=1}^{l}\sum_{r=q}^{l}\prod_{i=1}^{m}((p,p-1)_i(r,r-q)_i) =$$

$$= \sum_{q=0}^{l} z^{q+1}\sum_{p=q+1}^{l}\prod_{i=1}^{m}(p,p-q-1)_i = \sum_{q=1}^{l} z^q \sum_{p=q}^{l}\prod_{i=1}^{m}(p,p-q)_i\,.$$

Thus by Lemma 8.2.4 c),

$$((\mu X)Z_1)(z) = \sum_{q=0}^{l} z^q \sum_{p=q}^{l}\prod_{i=1}^{m}(p,p-q)_i - \prod_{i=1}^{m} A_{n+i} +$$

$$+\sum_{r=0}^{l} X_r \prod_{i=1}^{m}(0,r)_i - \sum_{q=1}^{l} z^q \sum_{p=q}^{l}\prod_{i=1}^{m}(p,p-q)_i =$$

$$= \sum_{p=0}^{l}\prod_{i=1}^{m}(p,p)_i - \prod_{i=1}^{m} A_{n+i} + \sum_{r=0}^{l} X_r \prod_{i=1}^{m}(0,r)_i = 1_E - \prod_{i=1}^{m} A_{n+i} + \sum_{p=0}^{l} X_p \prod_{i=1}^{m}(0,p)_i\,.$$

By Lemma 8.2.4 a),b), for $z\in\mathbb{T}$,

$$(Y_1(\mu X)Z_1)(z) = 1_E - \prod_{i=1}^{m} A_{n+i} + \sum_{p=0}^{l} X_p \prod_{i=1}^{m}(0,p)_i - \sum_{p=1}^{l} X_p \prod_{i=1}^{m}(0,p)_i +$$

$$+\sum_{p=1}^{l} X_p \prod_{i=1}^{m}((0,p)_i A_{n+i}) - \sum_{p=1}^{l}\sum_{q=0}^{l} X_p X_q \prod_{i=1}^{m}((0,p)_i(0,q)_i) =$$

$$= 1_E - \prod_{i=1}^{m} A_{n+i} + X_0 \prod_{i=1}^{m} (0,0)_i = 1_E - \prod_{i=1}^{m} A_{n+i} + X \prod_{i=1}^{m} A_{n+i} \,.$$

Since $1_E - \prod_{i=1}^{m} A_{n+i} + X^{-1} \prod_{i=1}^{m} A_{n+i}$ is the inverse of $Y_1(\mu X)Z_1$ it follows that $Y_1(\mu X)Z_1$ and μX are invertible, i.e. they belong to $\mathscr{C}(\mathbb{T}, GL(F_{n+m}))$. Thus for every $s \in [0,1]$, $Y_s(\mu X)Z_s \in \mathscr{C}(\mathbb{T}, GL(F_{n+m}))$. Let $z \in \mathbb{T}$ and let

$$[0,1] \longrightarrow GL(F_n), \quad s \longmapsto x_s$$

be a continuous map with $x_0 = X(z)$ and $x_1 = 1_E$. Since $1_E - \prod_{i=1}^{m} A_{n+i} + x_s^{-1} \prod_{i=1}^{m} A_{n+i}$ is the inverse of $1_E - \prod_{i=1}^{m} A_{n+i} + x_s \prod_{i=1}^{m} A_{n+i}$ for every $s \in [0,1]$ it follows that the map

$$[0,1] \longrightarrow GL(F_{n+m}), \quad s \longmapsto 1_E - \prod_{i=1}^{m} A_{n+i} + x_s \prod_{i=1}^{m} A_{n+i}$$

is well-defined and it is a homotopy from $(Y_1(\mu X)Z_1)(z)$ to 1_E i.e. $Y_1(\mu X)Z_1 \in \mathscr{C}(\mathbb{T}, GL_0(F_{n+m}))$ and $Y_1(\mu X)Z_1 \in Pol(n+m,l)$. By the above, for every $s \in [0,1]$, $Y_s(\mu X)Z_s \in \mathscr{C}(\mathbb{T}, GL_0(F_{n+m}))$, so $Y_s(\mu X)Z_s \in Pol(n+m, 2l+1)$. Hence μX is homotopic to $X\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$ in $Pol(n+m, 2l+1)$ and $\mu X \in Lin(n+m)$.

In order to prove the last assertion remark that there is a $P \in Pr\, F_n$ with $X = \widetilde{P} = (1_E - P) + zP$. Then $m = l = 1$, $a_0 = 1_E - P$, $a_1 = P$, $X_1 = a_0 = P$,

$$\mu X = 1_E - PA_{n+1} + PC_{n+1}^{*} - zC_{n+1} \,,$$

and for every $s \in [0,1]$,

$$Y_s = 1_E - sPC_{n+1}^{*} \,, \qquad Z_s := 1_E + szC_{n+1} \,, \qquad Y_s(\mu X)Z_s \in Lin(n+1) \,.$$

Thus μX is homotopic to $Y_1(\mu X)Z_1$ in $Lin(n+1)$. ■

8.3 The Periodicity

Throughout this section F denotes a full E-C*-algebra, $m,n \in \mathbb{N}$, and $l := 2^m - 1$.

LEMMA 8.3.1 *If $X \in \mathscr{C}(\mathbb{T}, GL(F_n))$ and $X(1) \in GL_{E_n}(F_n)$ then*

$$X \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n)) \,.$$

Let $\theta \in [0, 2\pi[$ and for every $s \in [0,1]$ put

$$Y_s : \mathbb{T} \longrightarrow GL(F_n)\,, \quad z \longmapsto X(e^{-is}z)\,.$$

Then $Y_0(e^{i\theta}) = X(e^{i\theta})$ and $Y_\theta(e^{i\theta}) = X(1)$ so $X(e^{i\theta})$ is homotopic to $X(1)$ in $GL(F_n)$. Thus $X(e^{i\theta}) \in GL_{E_n}(F_n)$ and $X \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$. ■

PROPOSITION 8.3.2 *The following are equivalent for every $X \in F_n$.*

a) $\widetilde{X} \in Lin(n)$.

b) $z \in \mathbb{T} \setminus \{1\} \Longrightarrow \widetilde{X}(z) \in GL(F_n)$.

c) $\widetilde{X}$ *is a generalized idempotent of* F_n ([4] Definition 11.2.8).

$a \Rightarrow b$ is trivial.

$b \Rightarrow a$. By Lemma 8.3.1, since $\widetilde{X}(1) = 1_E$, $\tilde{X} \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$ so $\tilde{X} \in Lin(n)$.

$b \Leftrightarrow c$. For $z \in \mathbb{T} \setminus \{1\}$,

$$\widetilde{X}(z) = (z-1)X + 1_E = (z-1)\left(X - \frac{1}{1-z}1_E\right)\,.$$

Since

$$\left\{ \frac{1}{1-z} \;\middle|\; z \in \mathbb{T} \setminus \{1\} \right\} = \left\{ \alpha \in \mathbb{C} \mid real(\alpha) = \frac{1}{2} \right\},$$

b) holds iff $X - \alpha 1_E$ is invertible for every $\alpha \in \mathbb{C}$ with $real(\alpha) = \frac{1}{2}$, which is equivalent to c). ■

LEMMA 8.3.3 *For $z \in \mathbb{T}$,*

$$zA_n + B_n \sim_h A_n + zB_n \quad \text{in} \quad Un\,E_n\,.$$

We have

$$(C_n+C_n^*)(zA_n+B_n)(C_n+C_n^*)=(zC_n+C_n^*)(C_n+C_n^*)=zB_n+A_n$$

and the assertion follows from Proposition 6.2.5 a). ■

LEMMA 8.3.4 *For $z\in\mathbb{T}$,*

$$z^l\prod_{i=1}^{m}A_{n+i}+\sum_{p=1}^{l}\prod_{i=1}^{m}(p,p)_i\sim_h\prod_{i=1}^{m}A_{n+i}+z\sum_{p=1}^{l}\prod_{i=1}^{m}(p,p)_i\quad\text{in}\quad Un\,E_{n+m}\,.$$

Let $k\in\mathbb{N}_l$ and let $j\in\mathbb{N}_m$ with $|k|_j=1$. By Lemma 8.3.3,

$$\begin{aligned}
&z^{l-k+1}\prod_{i=1}^{m}A_{n+i}+z\sum_{p=1}^{k-1}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k}^{l}\prod_{i=1}^{m}(p,p)_i=\\
&=\left(z^{l-k}\prod_{i=1}^{m}A_{n+i}+\prod_{i=1}^{m}(k,k)_i\right)(zA_{n+j}+(k,k)_j)+\\
&\qquad+z\sum_{p=1}^{k-1}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k+1}^{l}\prod_{i=1}^{m}(p,p)_i\sim_h\\
&\sim_h\left(z^{l-k}\prod_{i=1}^{m}A_{n+i}+\prod_{i=1}^{m}(k,k)_i\right)(A_{n+j}+z(k,k)_j)+\\
&\qquad+z\sum_{p=1}^{k-1}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k+1}^{l}\prod_{i=1}^{m}(p,p)_i=\\
&=z^{l-k}\prod_{i=1}^{m}A_{n+i}+z\sum_{p=1}^{k}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k+1}^{l}\prod_{i=1}^{m}(p,p)_i
\end{aligned}$$

in $Un\,E_{n+m}$. The assertion follows now by induction on $k\in\mathbb{N}_l$. ■

LEMMA 8.3.5 *Let $P,Q\in Pr\,F_n$.*

a) For every $z\in\mathbb{T}$,

$$\overbrace{P\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}(z)=$$

$$= \widetilde{P}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) .$$

b) If (with the identification of Lemma 8.1.1 d)*)*

$$\widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m},$$

then

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \sim_h$$

$$\sim_h \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m} .$$

a) We have

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z) =$$

$$= zP\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) + \prod_{i=1}^{m} A_{n+i} +$$

$$+ \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) - P\left(\prod_{i=1}^{m} A_{n+i}\right) - \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) =$$

$$= \widetilde{P}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) .$$

b) Let

$$[0,1] \longrightarrow Un\left(\overbrace{SF}^{\vee}\right)_{n+m}, \quad s \longmapsto U_s$$

be a continuous map with

$$U_0 = \widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) ,$$

$$U_1 = \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) .$$

Put $U'_s := U_s\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$ for every $s \in [0,1]$. Then $s \mapsto U'_s$ is a continuous path in $Un\left(\overbrace{SF}^{\vee}\right)_{n+m}$ and by a),

$$U'_0 = U_0\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) =$$

$$= \widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) =$$

$$= \overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z) ,$$

$$U'_1 = \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z) . \qquad \blacksquare$$

PROPOSITION 8.3.6

a) If $U \in Un\left(\overbrace{SF}^{\vee}\right)_n$ then there are $k, m \in \mathbb{N}$ and $P \in Pr\, F_{n+m}$ such that (with the identification of Lemma 8.1.1 d)*)*

$$(z^k U)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h \widetilde{P} \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m} .$$

b) Let $P, Q \in Pr\, F_n$ with $\widetilde{P} \sim_h \widetilde{Q}$ in $Un\left(\overbrace{SF}^{\vee}\right)_n$. Then there is an $m \in \mathbb{N}$ such that

$$P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \quad \text{in} \quad Pr\, F_{n+m} .$$

a) By Proposition 8.2.2 a), there are $k,m \in \mathbb{N}$, $k < 2^m$, and $X \in Pol(n,l)$ such that $z^k U$ is homotopic to X in $\mathscr{C}(\mathbb{T}, GL_E(F_n))$. By Proposition 8.2.6, there is a $Y \in Lin(n+m)$ with

$$X\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h Y \quad \text{in} \quad Pol(n+m, 2l+1)\,.$$

By [4] Lemma 11.2.12 (i), there is a $P \in Pr\, F_{n+m}$ with $Y \sim_h \widetilde{P}$ in $Lin(n+m)$. Thus

$$(z^k U)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h X\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h Y \sim_h \widetilde{P}$$

in $\mathscr{C}(\mathbb{T}, GL_E(F_{n+m}))$. By [4] Proposition 2.1.8 (iii) and the identification of Lemma 8.1.1 d),

$$(z^k U)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h \widetilde{P} \quad \text{in} \quad Un\left(\widecheck{SF}\right)_{n+m}\,.$$

b) By Proposition 8.2.2 b), there are $k,m \in \mathbb{N}$, $k < 2^m$, such that $z^k\widetilde{P} \sim_h z^k\widetilde{Q}$ in $Pol(n,l)$. By Lemma 8.3.4 and Lemma 8.2.4 c),

$$z^l\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h \left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$$

in $Un\, E_{n+m}$. By Lemma 8.3.5 a),

$$\widetilde{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}\,(z) =$$

$$= \left(\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right) \times$$

$$\times \left(\widetilde{P}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right) \sim_h$$

$$\sim_h \left(z^l\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right)\left(\widetilde{P}(z) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right) =$$

$$= z^l \widetilde{P}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h z^l \widetilde{Q}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z)$$

in $Pol(n+m,l)$. By Proposition 8.2.6,

$$\widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) =$$

$$= \overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h \mu\left(\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}\right) \sim_h$$

$$\sim_h \mu\left(\overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}\right) \sim_h$$

$$\sim_h \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$$

in $Lin(n+m)$. By Lemma 8.3.5 a),

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} = \widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) = \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}$$

in $Lin(n+m)$. The assertion follows now from [4] Lemma 11.2.12 (ii). ■

THEOREM 8.3.7 *The Bott map is bijective.*

Step 1 Surjectivity

Let $a \in K_1(SF)$. There are $n \in \mathbb{N}$ and $U \in Un\left(\overbrace{SF}^{\vee}\right)_n$ with $a = [U]_1$. By Proposition 8.3.6 a), there are $m, p \in \mathbb{N}$, $p \geq n$, and $P \in Pr\, F_{p+m}$ such that

$$(z^l U)\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right) \sim_h \widetilde{P} \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{p+m}.$$

By Lemma 8.3.4 and Lemma 8.2.4 c),

$$\overbrace{1_E - \prod_{i=1}^{m} A_{p+i}}^{\sim} = z\left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + \left(\prod_{i=1}^{m} A_{p+i}\right) \sim_h$$

$$\sim_h \left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + z^l\left(\prod_{i=1}^{m} A_{p+i}\right) \quad \text{in} \quad Un\, E_{p+m}$$

so by Proposition 7.1.3 and Proposition 8.1.4,

$$\beta_F\left([P]_0 - \left[1_E - \prod_{i=1}^{m} A_{p+i}\right]_0\right) = [\widetilde{P}]_1 - \left[\overbrace{1_E - \prod_{i=1}^{m} A_{p+i}}^{\sim}\right]_1 =$$

$$= \left[(z^l U)\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right)\right]_1 -$$

$$- \left[\left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + z^l\left(\prod_{i=1}^{m} A_{p+i}\right)\right]_1 =$$

$$= \left[\left((z^l U)\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right)\right) \times\right.$$

$$\left.\times \left(\left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + z^l\left(\prod_{i=1}^{m} A_{p+i}\right)\right)^*\right]_1 =$$

$$= \left[U\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right)\right]_1 = [U]_1 = a\,.$$

Step 2 Injectivity

Let $a \in K_0(F)$ with $\beta_F a = 0$. By Proposition 6.1.5 d), there are $P, Q \in Pr\, F_n$, $PQ = 0$, such that $a = [P]_0 - [Q]_0$. Then $[\widetilde{P}]_1 = [\widetilde{Q}]_1$, so $U := \tilde{P}\tilde{Q}^* \in un_{E_n}\overbrace{SF}^{\vee}$. Then

$$U = ((z-1)P + 1_E)((\bar{z}-1)Q + 1_E)) = (z-1)P + (\bar{z}-1)Q + 1_E\,, \qquad U(1) = 1_E\,.$$

By Proposition 7.1.3, there is an $m \in \mathbb{N}$ such that

$$V := U\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) = \tau^F_{n+m,n} U \in Un_{E_{n+m}}\left(\overbrace{SF}^{\vee}\right)_{n+m}.$$

Then there is a $W \in Un\, E_{n+m}$ with $V \sim_h W$ in $Un\left(\overbrace{SF}^{\vee}\right)_{n+m}$. By the above,

$$W = W(1) \sim_h V(1) = 1_E\,, \qquad V \sim_h 1_E \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m}.$$

By Proposition 7.1.3,

$$\widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) = \tau^F_{n+m,n}\tilde{P} = (\tau^F_{n+m,n}U)(\tau^F_{n+m,n}\tilde{Q}) =$$

$$= V(\tau^F_{n+m,n}\tilde{Q}) \sim_h \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m},$$

so by Proposition 8.3.5 b),

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \sim_h$$

$$\sim_h \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m}.$$

Put

$$P' := P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right),$$

$$Q' := Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right).$$

By Proposition 8.3.6 b), there are $m', p' \in \mathbb{N}$ such that

$$P'\left(\prod_{j=1}^{m'} A_{p'+j}\right) + \left(1_E - \prod_{j=1}^{m'} A_{p'+i}\right) \sim_h$$

$$\sim_h \; Q'\left(\prod_{j=1}^{m'} A_{p'+i}\right) + \left(1_E - \prod_{j=1}^{m'} A_{p'+i}\right) \quad \text{in} \quad Pr\, F_{p'+m'} .$$

It follows successively

$$\left[P'\prod_{j=1}^{m'} A_{p'+j}\right]_0 = \left[Q'\prod_{j=1}^{m'} A_{p'+j}\right]_0 ,$$

$$\left[P\left(\prod_{i=1}^{m} A_{n+i}\right)\left(\prod_{j=1}^{m'} A_{p'+j}\right)\right]_0 = \left[Q\left(\prod_{i=1}^{m} A_{n+i}\right)\left(\prod_{j=1}^{m'} A_{p'+j}\right)\right]_0 ,$$

$$[P]_0 = [Q]_0 , \qquad a = [P]_0 - [Q]_0 = 0 .$$

■

Remark. By Theorem 8.3.7 and Proposition 8.1.5 c), the functor K_0 is determined by the functor K_1.

COROLLARY 8.3.8 (The six-term sequence) *Let*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$.

a) The sequence

$$0 \longrightarrow SF \xrightarrow{S\varphi} SG \xrightarrow{S\psi} SH \longrightarrow 0$$

is exact. Let

$$\delta_2 : K_1(SH) \longrightarrow K_0(SF)$$

be its associated index map (Corollary 7.2.3) *and put* (Proposition 8.1.5, Theorem 7.3.2)

$$\delta_0 := \theta_F^{-1} \circ \delta_2 \circ \beta_H : K_0(H) \longrightarrow K_1(F) .$$

We call δ_0 *and* δ_1 **the six-term index maps**. *If we denote by* $\bar{\delta}_0$ *the corresponding six-term index map associated to the exact sequence in* $\mathfrak{M}_E$ *(with obvious notation)*

$$0 \longrightarrow SF \xrightarrow{\varphi} CF \xrightarrow{\psi} F \longrightarrow 0$$

then $\bar{\delta}_0 = \beta_F$.

b) *The* **six-term sequence**

$$\begin{tikzcd} K_0(F) \arrow[r, "K_0(\varphi)"] & K_0(G) \arrow[r, "K_0(\psi)"] & K_0(H) \arrow[d, "\delta_0"] \\ K_1(H) \arrow[u, "\delta_1"] & K_1(G) \arrow[l, "K_1(\psi)"] & K_1(F) \arrow[l, "K_1(\varphi)"] \end{tikzcd}$$

is exact.

c) *If* F *(resp.* H*) is K-null (e.g. homotopic to* $\{0\}$*) then* $K_i(G) \xrightarrow{K_i(\psi)} K_i(H)$ *(resp.* $K_i(F) \xrightarrow{K_i(\varphi)} K_i(G)$*) is a group isomorphism for every* $i \in \{0,1\}$.

d) *If* G *is K-null (e.g. homotopic to* $\{0\}$*) then*

$$K_0(H) \xrightarrow{\delta_0} K_1(F)\,, \qquad\qquad K_1(H) \xrightarrow{\delta_1} K_0(F)$$

are group isomorphisms.

e) *If* φ *is K-null (e.g. factorizes through null) then the sequences*

$$0 \longrightarrow K_0(G) \xrightarrow{K_0(\psi)} K_0(H) \xrightarrow{\delta_0} K_1(F) \longrightarrow 0\,,$$

$$0 \longrightarrow K_1(G) \xrightarrow{K_1(\psi)} K_1(H) \xrightarrow{\delta_1} K_0(F) \longrightarrow 0$$

are exact.

f) *If* ψ *is K-null (e.g. factorizes through null) then the sequences*

$$0 \longrightarrow K_0(H) \xrightarrow{\delta_0} K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \longrightarrow 0\,,$$

$$0 \longrightarrow K_1(H) \xrightarrow{\delta_1} K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \longrightarrow 0$$

are exact.

g) *The six-term index maps of a split exact sequence are equal to* 0.

a) is easy to see.

b) By Theorem 8.3.7, β_H is an isomorphism. By Theorem 7.2.9, the sequences

$$K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \xrightarrow{K_1(\psi)} K_1(H) \xrightarrow{\delta_1} K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \xrightarrow{K_0(\psi)} K_0(H)\,,$$

$$K_1(SG) \xrightarrow{K_1(S\psi)} K_1(SH) \xrightarrow{\delta_2} K_0(SF) \xrightarrow{K_0(S\varphi)} K_0(SG)$$

are exact. By Proposition 8.1.5 c) and Proposition 7.3.8, the diagrams

$$\begin{tikzcd} K_0(G) \arrow[r, "K_0(\psi)"] \arrow[d, "\beta_G"'] & K_0(H) \arrow[d, "\beta_H"] \\ K_1(SG) \arrow[r, "K_1(S\psi)"'] & K_1(SH) \end{tikzcd} \qquad \begin{tikzcd} K_1(F) \arrow[r, "K_1(\varphi)"] \arrow[d, "\theta_F"'] & K_1(G) \arrow[d, "\theta_G"] \\ K_0(SF) \arrow[r, "K_0(S\varphi)"'] & K_0(SG) \end{tikzcd}$$

are commutative. It follows

$$\delta_0 \circ K_0(\psi) = \theta_F^{-1} \circ \delta_2 \circ \beta_H \circ K_0(\psi) = \theta_F^{-1} \circ \delta_2 \circ K_1(S\psi) \circ \beta_G = 0\,,$$

$Im\, K_0(\psi) \subset Ker\, \delta_0$. Let $a \in Ker\, \delta_0$. Then $\delta_2 \beta_H a = \theta_F \delta_0 a = 0$, so there is a $b \in K_1(SG)$ with $K_1(S\psi)\, b = \beta_H a$. It follows

$$a = \beta_H^{-1} K_1(S\psi)\, b = K_0(\psi)\, \beta_G^{-1} b \in Im\, K_0(\psi)\,, \qquad Ker\, \delta_0 \subset Im\, K_0(\psi)\ .$$

c) The assertion follows immediately from b). By Proposition 7.1.8 e), a null-homotopic E-C*-algebra is K-null.

d) The proof is similar to the proof of c).

e) and f) follow from b) and Proposition 7.1.8 f).

g) By Proposition 6.2.9 and Corollary 7.3.9 (with the notation of b)) $K_0(\varphi)$ and $K_1(\varphi)$ are injective and $K_0(\psi)$ and $K_1(\psi)$ are surjective and the assertion follows from b). ■

COROLLARY 8.3.9 *Let us consider the following commutative diagram in* $\mathfrak{M}_E$

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & G \arrow[r, "\psi"] \arrow[d, "\alpha"'] & H \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & F' \arrow[r, "\varphi'"'] & G' \arrow[r, "\psi'"'] & H' \arrow[r] & 0\,, \end{tikzcd}$$

where the horizontal lines are exact.

a) **(Commutativity of the six-term index maps)** *The diagrams (with obvious notation)*

$$\begin{array}{ccc} K_1(H) & \xrightarrow{\delta_1} & K_0(F) \\ {\scriptstyle K_1(\beta)}\downarrow & & \downarrow{\scriptstyle K_0(\gamma)} \\ K_1(H') & \xrightarrow[\delta_1']{} & K_0(F') \end{array} \qquad \begin{array}{ccc} K_0(H) & \xrightarrow{\delta_0} & K_1(F) \\ {\scriptstyle K_0(\beta)}\downarrow & & \downarrow{\scriptstyle K_1(\gamma)} \\ K_0(H') & \xrightarrow[\delta_0']{} & K_1(F') \end{array}$$

are commutative. If $K_i(F) = K_i(F')$, $K_i(H) = K_i(H')$, and $K_i(\beta)$ and $K_i(\gamma)$ are the identity maps for all $i \in \{0,1\}$ then $\delta_i = \delta_i'$ for all $i \in \{0,1\}$.

b) *The diagram (with obvious notation)*

$$\begin{array}{ccccccccc} K_0(F) & = & K_0(F) & \xrightarrow{K_0(\varphi)} & K_0(G) & \xrightarrow{K_0(\psi)} & K_0(H) & = & K_0(H) \\ =\downarrow & & {\scriptstyle K_0(\gamma)}\downarrow & & {\scriptstyle K_0(\alpha)}\downarrow & & \downarrow{\scriptstyle K_0(\beta)} & & \downarrow = \\ K_0(F) & \xrightarrow{K_0(\gamma)} & K_0(F') & \xrightarrow{K_0(\varphi')} & K_0(G') & \xrightarrow{K_0(\psi')} & K_0(H') & \xleftarrow{K_0(\beta)} & K_0(H) \\ {\scriptstyle \delta_1}\uparrow & & {\scriptstyle \delta_1'}\uparrow & & & & \downarrow{\scriptstyle \delta_0'} & & \downarrow{\scriptstyle \delta_0} \\ K_1(H) & \xrightarrow[K_1(\beta)]{} & K_1(H') & \xleftarrow[K_1(\psi')]{} & K_1(G') & \xleftarrow[K_1(\varphi')]{} & K_1(F') & \xleftarrow[K_1(\gamma)]{} & K_1(F) \\ =\uparrow & & {\scriptstyle K_1(\beta)}\uparrow & & {\scriptstyle K_1(\alpha)}\uparrow & & \uparrow{\scriptstyle K_1(\gamma)} & & \uparrow = \\ K_1(H) & = & K_1(H) & \xleftarrow[K_1(\psi)]{} & K_1(G) & \xleftarrow[K_1(\varphi)]{} & K_1(F) & = & K_1(F) \end{array}$$

is commutative.

a) The commutativity of the first diagram was proved in Proposition 7.2.4. By Proposition 7.3.8, the diagram

$$\begin{array}{ccc} K_1(F) & \xrightarrow{K_1(\gamma)} & K_1(F') \\ {\scriptstyle \theta_F}\downarrow & & \downarrow{\scriptstyle \theta_{F'}} \\ K_0(SF) & \xrightarrow[K_0(S\gamma)]{} & K_0(SF') \end{array}$$

is commutative. By Proposition 7.2.4, the diagram

$$\begin{array}{ccc} K_1(SH) & \xrightarrow{\delta_2} & K_0(SF) \\ {\scriptstyle K_1(S\beta)}\downarrow & & \downarrow{\scriptstyle K_0(S\gamma)} \\ K_1(SH') & \xrightarrow[\delta_2']{} & K_0(SF') \end{array}$$

is commutative, where δ_2 and δ_2' are defined in Corollary 8.3.8 a). By Proposition 8.1.5 c), the diagram

$$\begin{array}{ccc} K_0(H) & \xrightarrow{K_0(\beta)} & K_0(H') \\ {\scriptstyle \beta_H}\downarrow & & \downarrow{\scriptstyle \beta_{H'}} \\ K_1(SH) & \xrightarrow[K_1(S\beta)]{} & K_1(SH') \end{array}$$

is commutative. It follows, by the definition of δ_0 (Corollary 8.3.8 a)),

$$K_1(\gamma)\circ\delta_0 = K_1(\gamma)\circ\theta_F^{-1}\circ\delta_2\circ\beta_H = \theta_{F'}^{-1}\circ K_0(S\gamma)\circ\delta_2\circ\beta_H =$$

$$= \theta_{F'}^{-1}\circ\delta_2'\circ K_1(S\beta)\circ\beta_H = \theta_{F'}^{-1}\circ\delta_2'\circ\beta_{H'}\circ K_0(\beta) = \delta_0'\circ K_0(\beta)\ .$$

b) follows from a) and Corollary 8.3.8 b). ■

Chapter 9

Variation of the Parameters

Throughout this chapter we endow $\{0,1\}$ with the structure of o group by identifying it with $\mathbf{Z}_2$.

9.1 Changing E

Let E' be a commutative unital C*-algebra, $\phi : E \longrightarrow E'$ a unital C*-homomorphism, and

$$f' : T \times T \longrightarrow Un\, E' , \quad (s,t) \longmapsto \phi f(s,t) .$$

Then $f' \in \mathscr{F}(T,E')$ and we may define E'_n with respect to f' for every $n \in \mathbb{N}$ like in Definition 5.0.2.

Let $n \in \mathbb{N}$ and put

$$C'_n := \sum_{t\in T_n} ((\phi C_{n,t}) \otimes id_K) V_t^{f'} \quad (\in E'_n) .$$

For every $s \in T_{n-1}$,

$$\sum_{t\in T_n} ((f(s^{-1}t,t) C_{n,ts^{-1}}) \otimes id_K) V_t^f = V_s^f C_n =$$

$$= C_n V_s^f = \sum_{t\in T_n} ((f(ts^{-1},s) C_{n,ts^{-1}}) \otimes id_K) V_t^f$$

so by [2] Theorem 2.1.9 a),

$$f(s^{-1}t,t) C_{n,s^{-1}t} = f(ts^{-1},s) C_{n,ts^{-1}}$$

for every $t \in T_n$. It follows

$$f'(s^{-1}t,t) C'_{n,s^{-1}t} = f'(ts^{-1},s) C'_{n,ts^{-1}} , \quad V_s^{f'} C'_n = C'_n V_s^{f'} , \quad C'_n \in (E'_{n-1})^c .$$

Thus $(C'_n)_{n\in\mathbb{N}}$ satisfies the conditions of Axiom 5.0.3 and we may construct a K-theory with respect to T, E', f', and $(C'_n)_{n\in\mathbb{N}}$, which we shall denote by K'.

Let F be an E'-C*-algebra. We denote by $\bar{F}$ or by $\Phi(F)$ the E-C*-algebra obtained by endowing the C*-algebra F with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha,x) \longmapsto (\phi\alpha)x .$$

If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_{E'}$, then $\bar{F} \xrightarrow{\bar{\varphi}} \bar{G}$ is a morphism in $\mathfrak{M}_E$, in a natural way.

Let F be an E'-C*-algebra and $n \in \mathbb{N}$. We put for every

$$X = \sum_{t\in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^f \in \check{\bar{F}}_n ,$$

$$X' := \sum_{t\in T_n} ((\phi\alpha_t, x_t)\otimes id_K)V_t^{f'} \quad (\in \check{F}_n)$$

and set

$$\phi_{F,n} : \check{\bar{F}}_n \longrightarrow \check{F}_n\,, \quad X \longmapsto X'\ .$$

Then $\phi_{F,n}$ is a unital C*-homomorphism (surjective or injective if ϕ is so ([2] Theorem 2.1.9 a))) such that $\phi_{F,n}(Un_{E_n}\check{\bar{F}}_n) \subset Un_{E'_n}\check{F}_n$ and $\phi_{F,n}\circ\sigma_n^{\bar{F}} = \sigma_n^F \circ \phi_{F,n}$. Thus we get for every $i \in \{0,1\}$ an associated group homomorphism $\Phi_{i,F} : K_i(\bar{F}) \longrightarrow K'_i(F)$.

Let E'' be a unital commutative C*-algebra, $\phi' : E' \longrightarrow E''$ a unital C*-homomorphism, and $\phi'' := \phi'\circ\phi$. Then we may do similar constructions for ϕ' and ϕ'' as we have done for ϕ. If F is an E''-C*-algebra, $\Phi'(F)$ and $\Phi''(F)$ the corresponding E'-C*-algebra and E-C*-algebra, respectively, then $\Phi''(F) = \Phi(\Phi'(F))$. If Φ'_i and Φ''_i are the equivalents of Φ_i with respect to ϕ' and ϕ'', respectively, then $\Phi''_{i,F} = \Phi'_{i,F}\circ\Phi_{i,\Phi'(F)}$ for every $i \in \{0,1\}$. If $E'' = E$ and $\phi'' = id_E$ then $C''_n = C_n$ for every $n \in \mathbb{N}$ and for every E-C*-algebra F, $\Phi''(F) = F$ and $\Phi''_{i,F} = id_{K_i(F)}$ for every $i \in \{0,1\}$. If in addition $\phi''' := \phi\circ\phi' = id_{E'}$ then $C'''_n = C'_n$ for every $n \in \mathbb{N}$ and for every E'-C*-algebra F, $\Phi'(\Phi(F)) = F$ and $\Phi'_{i,\Phi(F)}\circ \Phi_{i,F} = id_{K'_i(F)}$ for every $i \in \{0,1\}$, i.e. the K-theory and the K'-theory "coincide".

Remark. Let $P \in Pr\,E$, $0 < P < 1_E$, and put

$$Pf : T\times T \longrightarrow Un\,PE\,, \quad (s,t) \longmapsto Pf(s,t)\ .$$

Then $Pf \in \mathscr{F}(T,PE)$ and we denote by PK the K-theory with respect to T, PE, Pf, and $(PC_n)_{n\in\mathbb{N}}$. Then for every E-C*-algebra F and $i \in \{0,1\}$

$$K_i(F) \approx ((PK)_i(PF)) \times (((1_E - P)K)_i((1_E - P)F))\ .$$

If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ then

$$P\varphi : PF \longrightarrow PG\,, \quad Px \longmapsto P\varphi x$$

is a morphism in $\mathfrak{M}_{PE}$ and

$$K_i(\varphi) = (PK)_i(P\varphi)\times((1_E - P)K)_i((1_E - P)\varphi)$$

for every $i \in \{0,1\}$.

PROPOSITION 9.1.1 *We use the above notation and assume $i \in \{0,1\}$.*

a) If $F \xrightarrow{\varphi} G$ *is a morphism in* $\mathfrak{M}_{E'}$ *then the diagram*

$$\begin{array}{ccc} K_i(\bar{F}) & \xrightarrow{K_i(\bar{\varphi})} & K_i(\bar{G}) \\ {\scriptstyle \Phi_{i,F}}\downarrow & & \downarrow{\scriptstyle \Phi_{i,G}} \\ K'_i(F) & \xrightarrow[K'_i(\varphi)]{} & K'_i(G) \end{array}$$

is commutative.

b) For every E'*-C*-algebra* F *the diagram*

$$\begin{array}{ccc} K_0(\bar{F}) & \xrightarrow{\beta_{\bar{F}}} & K_1(\overline{SF}) \\ {\scriptstyle \Phi_{0,F}}\downarrow & & \downarrow{\scriptstyle \Phi_{1,SF}} \\ K'_0(F) & \xrightarrow[\beta'_F]{} & K'_1(SF)\,, \end{array}$$

is commutative, where β'_F *denotes the Bott map in the* K'*-theory.*

c) If

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_{E'}$ *then the diagram*

$$\begin{array}{ccc} K_1(\bar{H}) & \xrightarrow{\delta_1} & K_0(\bar{F}) \\ {\scriptstyle \Phi_{1,H}}\downarrow & & \downarrow{\scriptstyle \Phi_{0,F}} \\ K'_1(H) & \xrightarrow[\delta'_1]{} & K'_0(F) \end{array}$$

is commutative, where δ'_1 *denotes the index maps associated to the above exact sequences in the* K'*-theory.*

a) For every $n \in \mathbb{N}$ and

$$X = \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^f \in \check{\bar{F}}_n\,,$$

$$\check{\varphi}_n \phi_{F,n} X = \sum_{t \in T_n} (((\phi\alpha_t), \varphi x_t) \otimes id_K) V_t^{f'} = \phi_{G,n} \check{\bar{\varphi}}_n X\,.$$

b) For every $n \in \mathbb{N}$ and $P \in Pr\,\check{\bar{F}}_n$,

$$\phi_{SF,n} \widetilde{P} = (\widetilde{P})' = \widetilde{P'} = \widetilde{\phi_{F,n} P}\,.$$

c) Let $n \in \mathbb{N}$ and $U \in Un\,\check{\check{H}}_{n-1}$. By Proposition 7.2.1 a), there are $V \in Un\,\check{\check{G}}_n$ and $P \in Pr\,\check{\check{F}}_n$ such that

$$\check{\check{\psi}}_n V = A_n U + B_n U^*, \qquad \check{\check{\varphi}}_n P = V A_n V^* .$$

Then

$$\check{\psi}_n \phi_{G,n} V = \phi_{H,n} \check{\check{\psi}}_n V = A'_n(\phi_{H,n-1}U) + B'_n(\phi_{H,n-1}U)^*,$$

$$\check{\varphi}_n \phi_{F,n} P = \phi_{G,n} \check{\check{\varphi}}_n P = (\phi_{G,n}V) A'_n (\phi_{G,n}V)^*$$

so by Corollary 7.2.3,

$$\delta'_1 \Phi_{1,H}[U]_1 = \delta'_1[\phi_{H,n-1}U]_1 = [\phi_{F,n}P]_0 = \Phi_{0,F}[P]_0 = \Phi_{0,F}\delta_1[U]_1$$

$$\delta'_1 \circ \Phi_{1,H} = \Phi_{0,F} \circ \delta_1 . \qquad \blacksquare$$

LEMMA 9.1.2 *Let F,G be C*-algebras, $\varphi : F \longrightarrow G$ a surjective C*-homomorphism, and*

$$\psi : \mathscr{C}([0,1],F) \longrightarrow \mathscr{C}([0,1],G), \quad x \longmapsto \varphi \circ x .$$

a) ψ is surjective.

b) Assume F unital and let $v \in Un\,\mathscr{C}([0,1],G)$ such that there is an $x \in Un\,F$ with $\varphi x = v(0)$. Then there is a $u \in Un\,\mathscr{C}([0,1],F)$ with $\psi u = v$ and $u(0) = x$.

a) Let y be an element of $\mathscr{C}([0,1],G)$ which is piecewise linear, i.e. there is a family

$$0 = s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$$

such that for every $i \in \mathbb{N}_{n-1}$ and $t \in [0,1]$,

$$y((1-t)s_i + ts_{i+1}) = (1-t)y(s_i) + ty(s_{i+1}) .$$

Since φ is surjective, there is a family $(x_i)_{i\in\mathbb{N}_n}$ in F with $\varphi x_i = y(s_i)$ for every $i \in \mathbb{N}_n$. Define $x : [0,1] \longrightarrow F$ by putting

$$x((1-t)s_i + ts_{i+1}) := (1-t)x_i + tx_{i+1}$$

for every $i \in \mathbb{N}_{n-1}$ and $t \in [0,1]$. For $i \in \mathbb{N}_{n-1}$ and $t \in [0,1]$,

$$(\psi x)((1-t)s_i + ts_{i+1}) = \varphi((1-t)x_i + tx_{i+1}) =$$

$$= (1-t)y(s_i) + ty(s_{i+1}) = y((1-t)s_i + ts_{i+1}),$$

so $\psi x = y$, $y \in Im\,\psi$. Since the set of elements of $\mathscr{C}([0,1],G)$, which are piecewise linear, is dense in $\mathscr{C}([0,1],G)$ and $Im\,\psi$ is closed (as C*-homomorphism), ψ is surjective.

b) Let

$$w : [0,1] \longrightarrow Un\,G, \quad s \longmapsto v(0)^* v(s) .$$

Then $w \in Un\,\mathscr{C}([0,1],G)$ and $w(0) = 1_G$. Put

$$w_t : [0,1] \longrightarrow Un\,G, \quad s \longmapsto w(st)$$

for every $t \in [0,1]$. Then

$$[0,1] \longrightarrow Un\,\mathscr{C}([0,1],G), \quad t \longmapsto w_t$$

is a continuous path with $w_1 = w$ and $w_0 = 1_{\mathscr{C}([0,1],G)}$. Thus

$$w \in Un_0\,\mathscr{C}([0,1],G) .$$

By a), ψ is surjective, so by [4] Lemma 2.1.7 (i), there is a $u' \in Un\,\mathscr{C}([0,1],F)$ with $\psi u' = w$. Put

$$u : [0,1] \longrightarrow Un\,F, \quad s \longmapsto xu'(0)^* u'(s) .$$

Then $u \in Un\,\mathscr{C}([0,1],F)$, $u(0) = x$, and

$$(\psi u)(s) = \varphi(u(s)) = \varphi(xu'(0)^* u'(s)) = \varphi(x)((\psi u')(0))^*((\psi u')(s)) =$$

$$= v(0)w(0)^* w(s) = v(0)1_G v(0)^* v(s) = v(s)$$

for every $s \in [0,1]$, i.e. $\psi u = v$. ∎

THEOREM 9.1.3 *$\Phi_{i,F}$ is a group isomorphism for every $i \in \{0,1\}$ and for every E'-C*-algebra F.*

By Proposition 9.1.1 b), $\Phi_{0,F} = (\beta'_F)^{-1} \circ \Phi_{1,SF} \circ \beta_{\check{F}}$, so it suffices to prove the assertion for $\Phi_{1,F}$ only. Let $n \in \mathbb{N}$ and $U \in Un\,\check{F}_n$. Put $V := U(\sigma_n^F U)^* \sim_1 U$. Since $\sigma_n^F V = 1_{E'}$, V has the form

$$V = \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^{f'}$$

with $\alpha_t = \delta_{1,t} 1_{E'}$ and $x_t \in F$ for every $t \in T_n$. If we put

$$W := \sum_{t \in T_n} ((\delta_{1,t} 1_E, x_t) \otimes id_K) V_t^f$$

then $\phi_{F,n} W = V$ and we get $\Phi_{1,F}[W]_1 = [V]_1 = [U]_1$, so $\Phi_{1,F}$ is surjective. Thus we have to prove the injectivity of $\Phi_{1,F}$ only.

Let $a \in Ker\, \Phi_{1,F}$. We have to prove $a = 0$. There are $n \in \mathbb{N}$ and

$$U := \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^f \in Un\, \check{\check{F}}_n$$

with $a = [U]_1$, where $(\alpha_t, x_t) \in \check{F}$ for every $t \in T_n$. Since $[U']_1 = \Phi_{1,F}[U]_1 = 0$, by Proposition 7.1.3, there is an $m \in \mathbb{N}$ such that

$$U_0' := \left(\prod_{i=1}^m A'_{n+i} \right) U' + \left(1_{E'} - \prod_{i=1}^m A'_{n+i} \right)$$

is homotopic in $Un\, \check{F}_{n+m}$ to a $U_1' \in Un\, E'_{n+m}$ $(\subset Un\, \check{F}_{n+m})$. Thus there is a continuous path

$$U' : [0,1] \longrightarrow Un\, \check{F}_{n+m}, \quad s \longmapsto U_s'.$$

Case 1 ϕ is injective

Put

$$W_s' := U_s' \sigma_{n+m}^F (U_s'^* U_0') \; (\in Un\, \check{F}_{n+m})$$

for every $s \in [0,1]$. Then

$$\sigma_{n+m}^F W_s' = \sigma_{n+m}^F U_0' = \phi_{F,n+m} \left(\left(\prod_{i=1}^m A_{n+i} \right) (\sigma_n^{\bar{F}} U) + \left(1_E - \prod_{i=1}^m A_{n+i} \right) \right)$$

for every $s \in [0,1]$. If we put

$$W_s' =: \sum_{t \in T_{n+m}} ((\beta_{s,t}, y_{s,t}) \otimes id_K) V_t^{f'},$$

where $(\beta_{s,t}, y_{s,t}) \in \check{F}$ for all $s \in [0,1]$ and $t \in T_n$, then

$$\sum_{t \in T_{n+m}} ((\beta_{s,t}, 0) \otimes id_K) V_t^{f'} = \sigma_{n+m}^F W_s' =$$

$$= \phi_{F,n+m}\left(\left(\prod_{i=1}^{m} A_{n+i}\right)\sum_{t\in T_n}((\alpha_t,0)\otimes id_K)V_t^f + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right)$$

and so by [2] Theorem 2.1.9 a), there is a (unique) family $(\gamma_t)_{t\in T_{n+m}}$ in E with $\beta_{s,t} = \phi\gamma_t$ for every $s \in [0,1]$ and $t \in T_{n+m}$. Since ϕ is injective, ϕ_{n+m} is also injective and $\phi_{n+m}(\check{\check{F}}_{n+m})$ may be identified with a unital C*-subalgebra of $\check{F}_{n+m}$. Thus

$$W : [0,1] \longrightarrow Un\,\check{\check{F}}_{n+m}\,, \quad s \longmapsto \sum_{t\in T_{n+m}}((\gamma_t, y_{s,t})\otimes id_K)V_t^f$$

is a continuous path in $Un\,\check{\check{F}}_{n+m}$ with $\phi_{F,n+m}W_s = W'_s$ for every $s \in [0,1]$. It follows

$$\phi_{F,n+m}W_0 = W'_0 = U'_0 = \phi_{F.,n+m}\left(\left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right),$$

$$\phi_{F,n+m}W_1 = W'_1 = U'_1\sigma^F_{n+m}(U'^*_1U'_0) = \sigma^F_{n+m}U'_0 \in \phi_{F,n+m}(Un\,E'_{n+m})\,.$$

Since ϕ is injective, $\phi_{F,n+m}$ is also injective and we get

$$\left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) = W_0\,,$$

$$\left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \in Un_{E_{n+m}}\check{\check{F}}_{n+m}\,, \qquad g = [U]_1 = 0\,.$$

Case 2 ϕ is surjective

We put

$$\bar{U}_0 := \left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \;(\in Un\,\check{\check{F}}_{n+m})\,.$$

Since ϕ is surjective, $\phi_{F,n+m}$ is also surjective ([2] Theorem 2.1.9 a)). Since

$$\phi_{F,n+m}\bar{U}_0 = U'_0$$

it follows from Lemma 9.1.2 b), that there is a continuous path

$$[0,1] \longrightarrow Un\,\check{\check{F}}_{n+m}, \quad s \longmapsto U_s$$

with $\phi_{F,n+m}U_s = U'_s$ for every $s \in [0,1]$ and $U_0 = \bar{U}_0$. Since $\phi_{F,n+m}U_1 = U'_1 \in Un\,E'_{n+m}$, we have $\bar{U}_0 \in Un_{E_{n+m}}\check{\check{F}}_{n+m}$ and $g = [U]_1 = [\bar{U}_0]_1 = 0$.

Case 3 ϕ is arbitrary

There are a unital commutative C*-algebra E'' and a unital C*-homomor-phisms $\phi' : E \longrightarrow E''$ and $\phi'' : E'' \longrightarrow E'$ such that ϕ' is surjective, ϕ'' is injective, and $\phi = \phi'' \circ \phi'$ and the assertion follows from the first two cases and the considerations from the begin of the section. ■

COROLLARY 9.1.4 *Let E', E'' be unital commutative C*-algebras such that $E = E' \times E''$ and*

$$\phi' : E \longrightarrow E', \quad (x', x'') \longmapsto x',$$
$$\phi'' : E \longrightarrow E'', \quad (x', x'') \longmapsto x''.$$

If F' is an E'-C-algebra and F'' is an E''-C*-algebra then the map (with obvious notation)*

$$K_i(\Phi'(F') \times \Phi''(F'')) \longrightarrow K_i'(F') \times K_i''(F''), \quad a \longmapsto (\Phi'_{i,F'} \times \Phi''_{i,F''})(\varphi_i a)$$

is a group isomorphism for every $i \in \{0,1\}$, where

$$\varphi_i : K_i(\Phi'(F') \times \Phi''(F'')) \longrightarrow K_i(\Phi'(F')) \times K_i(\Phi''(F''))$$

is the canonical group isomorphism (Product Theorem (Corollary 6.2.10 b), Proposition 7.3.3 b)). ■

COROLLARY 9.1.5 *If $f(s,t) \in \mathbb{C}$ for all $s,t \in T$ and $C_n \in \mathbb{C}_n$ for all $n \in \mathbb{N}$and if $K^{\mathbb{C}}$ denotes the K-theory with respect to T, $\mathbb{C}$, f, and $(C_n)_{n\in\mathbb{N}}$ then $K_i(E) = K_i^{\mathbb{C}}(\mathscr{C}(\Omega, \mathbb{C}))$ for all $i \in \{0,1\}$, where Ω denotes the spectrum of E.* ■

PROPOSITION 9.1.6 *If F is an E'-C*-algebra then the map*

$$\varphi : E \times \Phi(F) \longrightarrow \overbrace{\Phi(F)}^{\vee}, \quad (\alpha, x) \longmapsto (\alpha, x - \phi\alpha)$$

is an E-C-isomorphism.*

For $(\alpha, x), (\beta, y) \in E \times \Phi(F)$ and $\gamma \in E$,

$$\varphi(\gamma(\alpha, x)) = \varphi(\gamma\alpha, (\phi\gamma)x) = (\gamma\alpha, (\phi\gamma)x - \phi(\gamma\alpha)) =$$

$$= (\gamma,0)(\alpha,x-\phi\alpha) = (\gamma,0)\varphi(\alpha,x)\,,$$

$$\varphi(\alpha,x)^* = \varphi(\alpha^*,x^*) = (\alpha^*,x^*-\phi\alpha^*) = (\varphi(\alpha,x))^*\,,$$

$$\varphi(\alpha,x)\varphi(\beta,y) = (\alpha,x-\phi\alpha)(\beta,y-\phi\beta) =$$

$$= (\alpha\beta,(\phi\alpha)y-\phi(\alpha\beta)+(\phi\beta)x-\phi(\alpha\beta)+xy-(\phi\beta)x-(\phi\alpha)y+\phi(\alpha\beta)) =$$

$$= (\alpha\beta,xy-\phi(\alpha\beta)) = \varphi(\alpha\beta,xy) = \varphi((\alpha,x)(\beta,y))\,,$$

so φ is an E-C*-homomorphism. The other assertions are easy to see. ■

9.2 Changing f

In all Propositions and Corollaries of this section we use the notation and assumptions of Example 5.0.4 and F denotes a C*-algebra.

LEMMA 9.2.1 *For every $n \in \mathbb{N}$ there is an $\varepsilon_n > 0$ such that for every $m \in \mathbb{N}$, $m \leq n$, and $\alpha \in Un\,\mathbb{C}$, $|\alpha - 1| < \varepsilon_n$, there is a unique $\beta_\alpha \in Un\,\mathbb{C}$, $|\beta_\alpha - 1| < \frac{1}{n}$, with $\beta_\alpha^m = \alpha$; moreover the map $\alpha \mapsto \beta_\alpha$ is continuous.*

If β,γ are distinct elements of $Un\,\mathbb{C}$ and $\beta^m = \gamma^m$ then

$$|\beta-\gamma| \geq |1-e^{\frac{2\pi i}{m}}| > \frac{1}{m} \geq \frac{1}{n}$$

and the assertion follows from the continuity of the corresponding branch of the map $\alpha \mapsto \sqrt[m]{\alpha}$. ■

DEFINITION 9.2.2 *For every finite group S we endow $\mathscr{F}(S,\mathbb{C})$ with the metric*

$$d_S(g,h) := \sup\{\,|g(s,t)-h(s,t)|\mid s,t \in S\}$$

for all $g,h \in \mathscr{F}(S,\mathbb{C})$.

Remark. $\mathscr{F}(S,\mathbb{C})$ endowed with the above metric is compact.

DEFINITION 9.2.3 *We put*

$$\Lambda(T,E) := \{\, \lambda : T \longrightarrow Un\, E \mid \lambda(1) = 1_E \,\}$$

and

$$\delta\lambda : T \times T \longrightarrow Un\, E\,, \quad (s,t) \longmapsto \lambda(s)\lambda(t)\lambda(st)^*$$

for every $\lambda \in \Lambda(T,E)$.

LEMMA 9.2.4 *Let S be a finite group and Ω a compact space.*

a) $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbf{C}) \,\}$ *is an open set of* $\mathscr{F}(S,\mathbf{C})$.

b) For every $\varepsilon' > 0$ there is an $\varepsilon > 0$ such that for all $g,h \in \mathscr{F}(S,\mathscr{C}(\Omega,\mathbf{C}))$, if

$$\|g(s,t) - h(s,t)\| < \varepsilon$$

for all $s,t \in S$ then there is a $\lambda \in \Lambda(S,\mathbf{C})$ such that $h = g\delta\lambda$ and $|\lambda(s) - 1| < \varepsilon'$ for all $s \in S$.

c) Let $g \in \mathscr{F}(S,\mathscr{C}(\Omega,\mathbf{C}))$ and $\phi : [0,1] \times \Omega \longrightarrow \Omega$ a continuous map. We put for every $u \in [0,1]$,

$$\phi_u := \phi(u,\cdot) : \Omega \longrightarrow \Omega\,,$$

$$g_u : S \times S \longrightarrow Un\, \mathbf{C}\,, \quad (s,t) \longmapsto g(s,t) \circ \phi_u\,.$$

Then $g_u \in \mathscr{F}(S,\mathscr{C}(\Omega,\mathbf{C}))$ for every $u \in [0,1]$ and there is a $\lambda \in \Lambda(S,\mathbf{C})$ with $g_1 = g_0\delta\lambda$.

a) By [3] Theorem 2.3.2 (iii),

$$\{\, \mathscr{S}(g) \mid g \in \mathscr{F}(S,\mathbf{C}) \,\} / \approx_{\mathscr{S}}$$

is finite. $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbf{C}) \,\}$ is obviously a closed subgroup of $\mathscr{F}(S,\mathbf{C})$. By the above and [2] Proposition 2.2.2 c), $\mathscr{F}(S,\mathbf{C})$ is the union of a finite family of closed pairwise disjoint sets homeomorphic to $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbf{C}) \,\}$, so $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbf{C}) \,\}$ is open.

b) By a), there is an $\varepsilon > 0$ such that for all $g',h' \in \mathscr{F}(S,\mathbf{C})$ with $d_S(g',h') < \varepsilon$ there is a $\lambda \in \Lambda(S,\mathbf{C})$ with $h' = g'\delta\lambda$. We may assume that

$$(1+\varepsilon)^{Card\, S} - 1 < \varepsilon_{Card\, S}\,,$$

where $\varepsilon_{Card\,S}$ was defined in Lemma 9.2.1.

We put for every $\omega \in \Omega$

$$g_\omega : S \times S \longrightarrow Un\,\mathbb{C}, \quad (s,t) \longmapsto (g(s,t))(\omega),$$

$$h_\omega : S \times S \longrightarrow Un\,\mathbb{C}, \quad (s,t) \longmapsto (h(s,t))(\omega).$$

Let $\omega \in \Omega$. By the above, there is a $\lambda_\omega \in \Lambda(S,\mathbb{C})$ with $g_\omega = h_\omega \delta \lambda_\omega$. Let $s \in S$ and let $n \in \mathbb{N}$ be the least natural number with $s^n = 1_S$. By [2] Proposition 3.4.1 c),

$$\lambda_\omega(s)^n = \prod_{j=1}^{n-1} (g_\omega(s^j,s)^* h_\omega(s^j,s)).$$

For every $j \in \mathbb{N}_{n-1}$,

$$\left\| 1_E - g(s^j,s)^* h(s^j,s) \right\| = \left\| g(s^j,s) - h(s^j,s) \right\| < \varepsilon,$$

$$\left\| \prod_{j=1}^{n-1} (g(s^j,s)^* h(s^j,s)) \right\| = \left\| \prod_{j=1}^{n-1} (1_E - (1_E - g(s^j,s)^* h(s^j,s))) \right\| < (1+\varepsilon)^n,$$

$$\left\| 1_E - \prod_{j=1}^{n-1} (g(s^j,s)^* h(s^j,s)) \right\| < (1+\varepsilon)^{n-1} - 1 < \varepsilon_{Card\,S}.$$

By Lemma 9.2.1, there is a unique $\gamma \in Un\,\mathbb{C}$ with

$$\gamma^n = \prod_{j=1}^{n-1} (g(s^j,s)^* h(s^j,s)), \qquad |\gamma - 1| < \frac{1}{Card\,S}.$$

For $\omega \in \Omega$, since $|1 - \lambda_\omega(s)| < \varepsilon_{Card\,S}$, we get $\lambda_\omega(s) = \gamma(s)$. So if we put

$$\lambda(s) : \Omega \longrightarrow \mathbb{C}, \quad \omega \longmapsto \gamma(s)$$

we have $\lambda \in \Lambda(S,\mathbb{C})$ and $g = h\delta\lambda$. By Lemma 9.2.1, we may choose ε in such a way that the inequality $|\lambda(s) - 1| < \varepsilon'$ holds for all $s \in S$.

c) By b), there is a family $(\lambda_i)_{i \in \mathbb{N}_n}$ in $\Lambda(S,\mathbb{C})$ and

$$0 = u_0 < u_1 < \cdots < u_{n-1} < u_n = 1$$

such that $g_{u_i} = g_{u_{i-1}} \delta \lambda_i$ for every $i \in \mathbb{N}_n$. By induction $g_0 \delta \left(\prod_{i=1}^{j} \lambda_i \right) = g_{u_j}$ for every $j \in \mathbb{N}_n$. Thus if we put $\lambda := \prod_{i=1}^{n} \lambda_i$ then $g_0 \delta \lambda = g_1$ ∎

Remark. Let $\lambda \in \Lambda(T,E)$ and $f' = f\delta\lambda$ $(\in \mathscr{F}(T,E))$. For every full E-C*-algebra F and $n \in \mathbb{N}$ we denote by F'_n the equivalent of F_n constructed with respect to f' instead of f (Definition 5.0.2). By [2] Proposition 2.2.2 $a_1 \Rightarrow a_2$, there is for every $n \in \mathbb{N}$ a unique E-C*-isomorphism $\varphi_n^F : F_n \longrightarrow F'_n$ such that for all $m,n \in \mathbb{N}$, $m < n$, the diagram

$$\begin{tikzcd} F_m \arrow[r, "\varphi_m^F"] \arrow[d] & F'_m \arrow[d] \\ F_n \arrow[r, "\varphi_n^F"'] & F'_n \end{tikzcd}$$

is commutative, where the vertical arrows are the canonical inclusions. We put $C'_n := \varphi_n^E C_n$ for evrey $n \in \mathbb{N}$. $(C'_n)_{n\in\mathbb{N}}$ satisfies the conditions of Axiom 5.0.3 with respect to f', so we can construct a K-theory with respect to T, E, f', and $(C'_n)_{n\in\mathbb{N}}$, which we shall denote by $K^{f'}$. If $m,n \in \mathbb{N}$, $m < n$, then the diagrams

$$\begin{tikzcd} F_m \arrow[r, "\rho_{n,m}^F"] \arrow[d, "\varphi_m^F"'] & F_n \arrow[d, "\varphi_n^F"] \\ F'_m \arrow[r, "\rho_{n,m}^{F'}"'] & F'_n \end{tikzcd} \qquad \begin{tikzcd} Un\, F_m \arrow[r, "\tau_{n,m}^F"] \arrow[d, "\varphi_m^F"'] & Un\, F_n \arrow[d, "\varphi_n^F"] \\ Un\, F'_m \arrow[r, "\tau_{n,m}^{F'}"'] & Un\, F'_n \end{tikzcd}$$

are commutative and so we get the isomorphisms

$$Pr\, F_{\rightarrow} \longrightarrow Pr\, F'_{\rightarrow}\,, \qquad\qquad un\, F_{\leftarrow} \longrightarrow un\, F'_{\leftarrow}\,.$$

By these considerations it can be followed that K and $K^{f'}$ coincide.

DEFINITION 9.2.5 *Let Ω be the spectrum of E, Γ a closed set of Ω, and F a C*-algebra. We denote by $\mathscr{C}(E;\Gamma,F)$ the E-C*-algebra obtained by endowing the C*-algebra $\mathscr{C}(\Gamma,F)$ with the structure of an E-C*-algebra by putting*

$$\alpha x : \Gamma \longrightarrow F\,, \quad \omega \longmapsto \alpha(\omega)x(\omega)$$

for all $(\alpha,x) \in E \times \mathscr{C}(\Gamma,F)$. If Ω' is an open set of Ω then the ideal and E-C-subalgebra*

$$\left\{\, x \in \mathscr{C}(E;\Omega,F) \mid x|(\Omega \setminus \Omega') = 0 \,\right\}$$

of $\mathscr{C}(E;\Omega,F)$ will be denoted $\mathscr{C}_0(E;\Omega',F)$.

By Tietze's theorem

$$0 \longrightarrow \mathscr{C}_0\left(E;\Omega',F\right) \xrightarrow{\varphi} \mathscr{C}\left(E;\Omega,F\right) \xrightarrow{\psi} \mathscr{C}\left(E;\Omega\setminus\Omega',F\right) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$, where φ denotes the inclusion map and

$$\psi : \mathscr{C}\left(E;\Omega,F\right) \longrightarrow \mathscr{C}\left(E;\Omega\setminus\Omega',F\right), \quad x \longmapsto x|(\Omega\setminus\Omega') .$$

PROPOSITION 9.2.6 *We denote by Ω the spectrum of E, by Γ a closed set of Ω, and by $\vartheta : [0,1]\times\Omega \longrightarrow \Omega$ a continuous map such that*

$$\omega\in\Omega \Longrightarrow \vartheta(0,\omega)=\omega,\ \vartheta(1,\omega)\in\Gamma$$

and $\vartheta(s,\omega)=\omega$ for all $s\in[0,1]$ and $\omega\in\Gamma$. We put $E':=\mathscr{C}\left(\Gamma,\mathbb{C}\right)$, $E'':=E$, $\vartheta_s:=\vartheta(s,\cdot)$ for every $s\in[0,1]$, and

$$\phi : E \longrightarrow E', \quad x\longmapsto x|\Gamma, \qquad\qquad \phi' : E'\longrightarrow E''=E, \quad x'\longmapsto x'\circ\vartheta_1 ,$$

$$f' : T\times T \longrightarrow Un\,E', \quad (s,t)\longmapsto \phi f(s,t)=f(s,t)|\Gamma,$$

$$f'' : T\times T \longrightarrow Un\,E'', \quad (s,t)\longmapsto \phi' f'(s,t)=f(s,t)\circ\vartheta_1 .$$

a) There is a $\lambda\in\Lambda(T,E)$ such that $f''=f\delta\lambda$ and the K-theories associated to f and f'' coincide (as formulated in the above Remark*). If Γ is a one-point set (i.e. Ω is contractible) then $f''(s,t)\in Un\,\mathbb{C}$ $(\subset Un\,E)$ for all $s,t\in T$.*

b) If we put

$$\psi : \mathscr{C}\left(E;\Omega,F\right) \longrightarrow \mathscr{C}\left(E;\Gamma,F\right), \quad x\longmapsto x|\Gamma$$

then $K_i(\mathscr{C}_0\left(E;\Omega\setminus\Gamma,F\right))=\{0\}$ and

$$K_i(\psi) : K_i(\mathscr{C}\left(E;\Omega,F\right)) \longrightarrow K_i(\mathscr{C}\left(E;\Gamma,F\right))$$

is a group isomorphism for every $i\in\{0,1\}$.

c) If Γ' is a compact subspace of $\Omega\setminus\Gamma$ then

$$K_i(\mathscr{C}_0\left(E;\Omega\setminus(\Gamma\cup\Gamma'),F\right)) \approx K_{i+1}(\mathscr{C}\left(E;\Gamma',F\right))$$

for all $i\in\{0,1\}$.

d) Let $\bar{\Gamma}$ be a closed set of Ω, $\bar{\varphi} : \mathscr{C}_0\left(E; \Omega \setminus (\Gamma \cup \bar{\Gamma}), F\right) \longrightarrow \mathscr{C}(E; \Omega, F)$ the inclusion map,

$$\bar{\psi} : \mathscr{C}_0(E; \Omega, F) \longrightarrow \mathscr{C}\left(E; \Gamma \cup \bar{\Gamma}, F\right), \quad x \longmapsto x|(\Gamma \cup \bar{\Gamma}),$$

and δ_0, δ_1 the corresponding maps from the six-term sequence associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0\left(E; \Omega \setminus (\Gamma \cup \bar{\Gamma}), F\right) \xrightarrow{\bar{\varphi}} \mathscr{C}(E; \Omega, F) \xrightarrow{\bar{\psi}} \mathscr{C}\left(E; \Gamma \cup \bar{\Gamma}, F\right) \longrightarrow 0$$

then the sequence

$$0 \longrightarrow K_i(\mathscr{C}(E; \Omega, F)) \xrightarrow{K_i(\psi)} K_i(\mathscr{C}\left(E; \Gamma \cup \bar{\Gamma}, F\right)) \xrightarrow{\delta_i}$$

$$\xrightarrow{\delta_i} K_{i+1}(\mathscr{C}_0\left(E; \Omega \setminus (\Gamma \cup \bar{\Gamma}), F\right)) \longrightarrow 0$$

is exact for every $i \in \{0, 1\}$.

a) By Lemma 9.2.4 c), for every $m \in \mathbb{N}$ there is a $\lambda_m \in \Lambda(S_m, E)$ with $f''|(S_m \times S_m) = g_m \delta \lambda_m$. We put

$$\lambda : T \longrightarrow Un\,E, \quad t \longmapsto \lambda_m(t) \quad \text{if} \quad t \in S_m .$$

Then

$$f''(s,t) = \prod_{m \in \mathbb{N}} (g_m \delta \lambda)(s_m, t_m) = (f \delta \lambda)(s,t)$$

for all $s, t \in T$, i.e. $f'' = f \delta \lambda$.

b) Let $n \in \mathbb{N}$ and $X \in \left(\overbrace{\mathscr{C}_0\left(E''; \Omega \setminus \Gamma, F\right)}^{\vee}\right)_n$. Then X has the form

$$X = \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^{f''},$$

where $\alpha_t \in E''$ and $x_t \in \mathscr{C}_0(E''; \Omega \setminus \Gamma, F)$ for all $t \in T_n$. We put

$$X_s := \sum_{t \in T_n} ((\alpha_t \circ \vartheta_s, x_t \circ \vartheta_s) \otimes id_K) V_t^{f''}$$

for every $s \in [0, 1]$. Then

$$[0,1] \longrightarrow \left(\overbrace{\mathscr{C}_0\left(E''; \Omega \setminus \Gamma, F\right)}^{\vee}\right)_n, \quad s \longmapsto X_s$$

is a continuous map, $X_0 = X$,

$$X_1 = \sum_{t \in T} ((\alpha_t \circ \vartheta_1, 0) \otimes id_K) V_t^{f''},$$

and

$$\left(\overbrace{\mathscr{C}_0 \left(E'' : \Omega \setminus \Gamma, F \right)}^{\vee} \right)_n \longrightarrow \left(\overbrace{\mathscr{C}_0 \left(E''; \Omega \setminus \Gamma, F \right)}^{\vee} \right)_n, \quad X \longmapsto X_s$$

is an E''-C*-homomorphism for every $s \in [0,1]$. Thus $K_i^{f''}(\mathscr{C}_0 (E''; \Omega \setminus \Gamma, F)) = \{0\}$. By a), $K_i(\mathscr{C}_0 (E; \Omega \setminus \Gamma, F)) = \{0\}$.

If $\varphi : \mathscr{C}_0 (E;\Omega \setminus \Gamma, f) \longrightarrow \mathscr{C} (E;\Omega, F)$ denotes the inclusion map then

$$0 \longrightarrow \mathscr{C}_0 (E; \Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C} (E; \Omega, F) \xrightarrow{\psi} \mathscr{C} (E; \Gamma, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ and the assertion follows from the six-term sequence (Corollary 8.3.8 c)).

c) If we put

$$F_1 := \mathscr{C}_0 \left(E; \Omega \setminus (\Gamma \cup \Gamma'), F \right), \quad F_2 := \mathscr{C}_0 (E; \Omega \setminus \Gamma, F), \quad F_3 := \mathscr{C} \left(E; \Gamma', F \right),$$

$$\varphi : F_1 \longrightarrow F_2, \quad x \longmapsto x,$$

$$\psi : F_2 \longrightarrow F_3, \quad x \longmapsto x|\Gamma'$$

then

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow{\psi} F_3 \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ and the assertion follows from b) and from the six-term sequence (Corollary 8.3.8 d)).

d) $\bar{\varphi}$ factorizes through $\mathscr{C}_0 (E; \Omega \setminus \Gamma, f)$ so by b), $K_i(\bar{\varphi}) = 0$ and the assertion follows from the six-term sequence Corollary 8.3.8 b). ■

COROLLARY 9.2.7 *We use the notation of* Proposition 9.2.6. *Let $\bar{\Omega}$ be a compact space and $\bar{\vartheta} : \Omega \longrightarrow \bar{\Omega}$ a continuous map such that the induced maps $\Omega \setminus (\Gamma \cup \Gamma') \to \bar{\Omega} \setminus \bar{\vartheta}(\Gamma \cup \Gamma')$, $\Gamma \to \bar{\vartheta}(\Gamma)$, and $\Gamma' \to \bar{\vartheta}(\Gamma')$ are homeomorphisms. If we put $\bar{E} := \mathscr{C} \left(\bar{\Omega}, \mathbb{C} \right)$ and*

$$\bar{\phi} : \bar{E} \longrightarrow E, \quad x \longmapsto x \circ \bar{\vartheta}$$

and take an $\bar{f} \in \mathscr{F}(T, \bar{E})$ *such that* $f(s,t) = \bar{\phi}\bar{f}(s,t)$ *for all* $s, t \in T$ *and a corresponding* $(\bar{C}_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} \bar{E}_n$ *then with the notation from the beginning of* section 9.1 (with E and $\bar{E}$ interchanged)

$$\bar{K}_i\left(\mathscr{C}_0\left(\bar{E}; \bar{\Omega}\setminus\bar{\vartheta}(\Gamma\cup\Gamma'), F\right)\right) \approx \bar{K}_{i+1}\left(\mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma'), F\right)\right),$$

for all $i \in \{0,1\}$*, where* $\bar{K}$ *denotes the K-theory associated to* T*,* $\bar{E}$*,* $\bar{f}$*, and* $(\bar{C}_n)_{n\in\mathbb{N}}$*. If in addition* Γ' *has the same property as* Γ *then*

$$\bar{K}_i\left(\mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma), F\right)\right) \approx \bar{K}_i\left(\mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma'), F\right)\right).$$

By our hypotheses,

$$\bar{\Phi}\left(\mathscr{C}_0\left(E; \Omega\setminus(\Gamma\cup\Gamma'), F\right)\right) \approx \mathscr{C}_0\left(\bar{E}; \bar{\Omega}\setminus\bar{\vartheta}(\Gamma\cup\Gamma'), F\right),$$

$$\bar{\Phi}(\mathscr{C}(E; \Gamma, F)) \approx \mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma), F\right), \qquad \bar{\Phi}\left(\mathscr{C}\left(E; \Gamma', F\right)\right) \approx \mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma'), F\right),$$

so by Proposition 9.2.6 b) and Theorem 9.1.3,

$$\bar{K}_i\left(\mathscr{C}_0\left(\bar{E}; \bar{\Omega}\setminus\bar{\vartheta}(\Gamma\cup\Gamma'), F\right)\right) \approx K_i\left(\mathscr{C}_0\left(E; \Omega\setminus(\Gamma\cup\Gamma'), F\right)\right) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}\left(E; \Gamma', F\right)\right) \approx \bar{K}_{i+1}(\mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma'), F\right)).$$

If the supplementary hypothesis is fulfilled then by Proposition 9.2.6 c) and Theorem 9.1.3,

$$\bar{K}_i\left(\mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma), F\right)\right) \approx K_i(\mathscr{C}(E; \Gamma, F)) \approx$$

$$\approx K_i\left(\mathscr{C}\left(E; \Gamma'), F\right)\right) \approx \bar{K}_i\left(\mathscr{C}\left(\bar{E}; \bar{\vartheta}(\Gamma'), F\right)\right).$$

■

COROLLARY 9.2.8 *Assume* $E = \mathscr{C}(\mathbb{T}, \mathbb{C})$.

a) If θ_1, θ_2, θ_3, $\theta_4 \in \mathbb{R}$ *such that* $\theta_1 \le \theta_2 < \theta_1 + 2\pi$*,* $\theta_3 \le \theta_4 < \theta_3 + 2\pi$ *then*

$$K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \,\middle|\, \theta_1 \le \theta \le \theta_2 \right\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \,\middle|\, \theta_3 \le \theta \le \theta_4 \right\}, F\right)\right)$$

for every $i \in \{0,1\}$.

b) Let $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 \leq \theta_2 < \theta_1 + 2\pi$ and let Γ be a closed set of

$$\mathbb{T} \setminus \left\{ e^{i\theta} \,\middle|\, \theta_2 < \theta < \theta_1 + 2\pi \right\}$$

such that $e^{i\theta_1} \in \Gamma$ and $e^{i\theta_2} \notin \Gamma$ if $e^{i\theta_1} \neq e^{i\theta_2}$. Then

$$K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}(E; \Gamma, F))$$

for every $i \in \{0,1\}$. Moreover

$$K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F)) \approx \begin{cases} K_{i+1}(\mathscr{C}(E; \{1\}, F))^{\Gamma} & \text{if} \quad F \text{ is finite} \\ \sum\limits_{n \in \mathbb{N}} K_{i+1}(\mathscr{C}(E; \{1\}, F)) & \text{if} \quad F \text{ is infinite} \end{cases} .$$

c) If Γ_1, Γ_2 are closed sets of $\mathbb{T}$, not equal to $\mathbb{T}$ and such that their cardinal numbers are equal if they are finite then

$$K_i(\mathscr{C}(E; \Gamma_1, F)) \approx K_i(\mathscr{C}(E; \Gamma_2, F))$$

for all $i \in \{0,1\}$.

a) We may assume $\theta_1 \leq \theta_3 < \theta_1 + 2\pi$. Put $\Omega' := [\theta_1, \sup(\theta_2, \theta_3)]$, $E' := \mathscr{C}(\Omega', \mathbb{C})$,

$$\vartheta : \Omega' \longrightarrow \mathbb{T}, \quad \alpha \longmapsto e^{i\alpha},$$

$$\phi : E \longrightarrow E', \quad x \longmapsto x \circ \vartheta .$$

Since it is possible to find an $f' \in \mathscr{F}(T, E')$ and a $(C'_n)_{n \in \mathbb{N}} \in \prod\limits_{n \in \mathbb{N}} E'_n$ with the desired properties, we get

$$K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \,\middle|\, \theta_1 \leq \theta \leq \theta_2 \right\}, F\right)\right) \approx K_i\left(\mathscr{C}\left(E; \{\mathrm{e}^{\mathrm{i}\theta_3}\}, F\right)\right) .$$

by Corollary 9.2.7. Thus

$$K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \,\middle|\, \theta_3 \leq \theta \leq \theta_4 \right\}, F\right)\right) \approx K_i\left(\mathscr{C}\left(E; \{\mathrm{e}^{\mathrm{i}\theta_3}\}, F\right)\right),$$

$$K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \,\middle|\, \theta_1 \leq \theta \leq \theta_2 \right\}, F\right)\right) \approx$$
$$\approx K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \,\middle|\, \theta_3 \leq \theta \leq \theta_4 \right\}, F\right)\right) .$$

b) If we put $\Omega' := [\theta_1, \theta_1 + 2\pi]$, $E' := \mathscr{C}(\Omega', \mathbb{C})$,

$$\vartheta : \Omega' \longrightarrow \mathbb{T}, \quad \alpha \longmapsto e^{i\alpha},$$

$$\phi : E \longrightarrow E', \quad x \longmapsto x \circ \vartheta ,$$

then the first assertion follows from Corollary 9.2.7. If Γ is finite then the last assertion follows now from a) (and Corollary 6.2.10 b) and Proposition 7.3.1 b)).

Assume now Γ infinite. Then $\Omega_0 := \mathbb{T} \setminus \Gamma$ is the union of a countable set of open intervals. Let Ξ be the set of finite such intervals ordered by inclusion and for every $\Theta \in \Xi$ let Ω_Θ be the union of the intervals of Θ and $\Gamma_\Theta := \mathbb{T} \setminus \Omega_\Theta$. By the above,

$$K_i(\mathscr{C}_0(E;\mathbb{T} \setminus \Gamma_\Theta, F)) \approx K_{i+1}(\mathscr{C}(E;\{1\},F))^\Theta$$

for every $\Theta \in \Xi$. We get an inductive system of E-modules with $\mathscr{C}_0(E;\mathbb{T} \setminus \Gamma, F)$ as inductive limit. By Theorem 6.2.12 and Theorem 7.3.6, $K_i(\mathscr{C}_0(E;\mathbb{T} \setminus \Gamma, F))$ is the inductive limit of $K_i(\mathscr{C}_0(E;\mathbb{T} \setminus \Gamma_\Theta, F))$ for Θ running through Ξ, which proves the assertion.

c) follows from b). ■

Remark. Let δ_0 and δ_1 be the group homomorphisms from the six-term sequence associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(E;\mathbb{T} \setminus \Gamma, F) \longrightarrow \mathscr{C}(E;\mathbb{T}, F) \longrightarrow \mathscr{C}(E;\Gamma, F) \longrightarrow 0 .$$

Then δ_0 and δ_1 do not coincide with the group isomorphism

$$K_i(\mathscr{C}_0(E;\mathbb{T} \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}(E;\Gamma, F))$$

from Corollary 9.2.8 b).

COROLLARY 9.2.9 *If Ω is a compact space such that $E = \mathscr{C}(\Omega \times \mathbb{T}, \mathbb{C})$ then*

$$K_i(\mathscr{C}_0(E;\Omega \times (\mathbb{T} \setminus \{1\}), F)) \approx K_{i+1}(\mathscr{C}(E;\Omega \times \{1\}, F))$$

for every $i \in \{0,1\}$. ■

COROLLARY 9.2.10 *If the spectrum of E is $\mathbb{B}_n$ for some $n \in \mathbb{N}$ then $K_i(\mathscr{C}_0(E;\mathbb{B}_n \setminus \{0\}, F)) = \{0\}$ and*

$$K_i(\mathscr{C}_0(E;\{\, \alpha \in \mathbb{R}^n \mid 0 < \|\alpha\| < 1 \,\}, F)) \approx K_{i+1}(\mathscr{C}(E;\mathbf{S}_{n-1}, F))$$

for every $i \in \{0,1\}$. ■

COROLLARY 9.2.11 *Let $(k_j)_{j\in J}$ be a finite family in $\mathbb{N}$, Ω' the topological sum of the family of balls $(\mathbb{B}_{k_j})_{j\in J}$, and Ω the compact space obtained from Ω' by identifying the centers of theses balls. If ω denotes the point of Ω obtained by this identification and S denotes the union of $(\mathbb{S}_{k_j-1})_{j\in J}$ in Ω and if $E=\mathscr{C}(\Omega,\mathbb{C})$ then*

$$K_i(\mathscr{C}_0(E;\Omega\setminus\{\omega\},F))=\{0\},$$

$$K_i(\mathscr{C}_0(E;(\Omega\setminus(\{\omega\}\cup S),F))\approx K_{i+1}(\mathscr{C}(E;S,F))$$

for every $i\in\{0,1\}$.

If we denote by $\vartheta:\Omega'\longrightarrow\Omega$ the quotient map, by Γ the subset of Ω' formed by the centers of the balls $(\mathbb{B}_{k_j})_{j\in J}$, and by Γ' the union of $(\mathbb{S}_{k_j-1})_{j\in J}$ ($\Gamma'\subset\Omega'$) then the assertions follow from Proposition 9.2.6 b), c) and Corollary 9.2.7. ■

LEMMA 9.2.12 *Let S be a finite group, $g\in\mathscr{F}(S,E)$, and Ω the spectrum of E.*

a) If there is an $\omega_0\in\Omega$ and a family $(\theta(s,t))_{s,t\in S}$ of selfadjoint elements of E such that

$$\theta(r,s)+\theta(rs,t)=\theta(r,st)+\theta(s,t),\quad g(s,t)=e^{i\theta(s,t)}(g(s,t)(\omega_0))$$

for all $r,s,t\in S$ then there is a $\lambda\in\Lambda(S,\mathbb{C})$ with $(g\delta\lambda)(s,t)=g(s,t)(\omega_0)$ for all $s,t\in S$.

b) If Ω is totally disconnected then there is a $\lambda\in\Lambda(S,E)$ such that

$$((g\delta\lambda)(s,t))(\Omega)$$

is finite for all $s,t\in S$.

a) For every $u\in[0,1]$ put

$$g_u:S\times S\longrightarrow Un\,E\,,\quad (s,t)\longmapsto e^{iu\theta(s,t)}(g(s,t)(\omega_0))\,.$$

Then

$$[0,1]\longrightarrow\mathscr{F}(S,E),\quad u\longmapsto g_u$$

is a continuous map with $g_1 = g$ and $g_0(s,t) = g(s,t)(\omega_0)$ for all $s,t \in S$. By Lemma 9.2.4 a),b), there are

$$0 = u_0 < u_1 < \cdots < u_{k-1} < u_k = 1$$

and a family $(\lambda_j)_{j\in\mathbb{N}_k}$ in $\Lambda(S,\mathbf{C})$ such that $g_{u_{j-1}} = g_{u_j}\delta\lambda_j$ for every $j \in \mathbb{N}_k$. We prove by induction that

$$g_{u_{l-1}} = g\prod_{j=l}^{k}\delta\lambda_j$$

for all $l \in \mathbb{N}_k$. This is obvious for $l = k$. Assume the identity holds for $l \in \mathbb{N}_k$, $l > 1$. Then

$$g\prod_{j=l-1}^{k}\delta\lambda_j = \left(g\prod_{j=l}^{k}\delta\lambda_j\right)\delta\lambda_{l-1} = g_{u_{l-1}}\delta\lambda_{l-1} = g_{u_{l-2}}\,,$$

which finishes the proof by induction. If we put

$$\lambda := \prod_{j=1}^{k}\lambda_j \in \Lambda(S,\mathbf{C})$$

then by the above

$$g\delta\lambda = g\prod_{j=1}^{k}\delta\lambda_j = g_0\,.$$

b) Let $\omega_0 \in \Omega$. Since Ω is totally disconnected and S is finite, by continuity, there is a clopen neighborhood Ω_0 of ω_0 and a family $(\theta(s,t))_{s,t\in S}$ in $Re\mathscr{C}(\Omega_0,\mathbf{C})$ such that

$$\theta(r,s)+\theta(rs,t) = \theta(r,st)+\theta(s,t)\,, \qquad g(s,t)|\Omega_0 = e^{i\theta(s,t)}(g(s,t)(\omega_0))$$

for all $r,s,t \in S$. By a), there is a $\lambda \in \Lambda(S,\mathbf{C})$ with

$$((g|\Omega_0)\delta\lambda)(s,t) = g(s,t)(\omega_0)$$

for all $s,t \in S$.

The assertion follows now from the fact that there is a finite partition $(\Omega_j)_{j\in J}$ of Ω with clopen sets such that Ω_j possesses the property of the above Ω_0 for every $j \in J$. ■

PROPOSITION 9.2.13 *If the spectrum of E is totally disconnected then there is a $\lambda \in \Lambda(T,E)$ such that $((f\delta\lambda)(s,t))(\Omega)$ is finite for all $s,t \in T$.*

By Lemma 9.2.12 b), for every $m \in \mathbb{N}$ there is a $\lambda_m \in \Lambda(S_m, E)$ such that $((g_m \delta \lambda_m)(s,t))(\Omega)$ is finite for all $s,t \in S_m$. If we put

$$\lambda : T \longrightarrow Un\, E\,, \quad t \longmapsto \lambda_m(t) \quad \text{if} \quad t \in S_m$$

then λ has the desired properties. ■

PROPOSITION 9.2.14 *Assume that T, f, and $(C_n)_{n\in\mathbb{N}}$ satisfy the conditions of* Example 5.0.4 *and of its* Remark 1 *and that the spectrum Ω of E is simply connected.*

a) There is a $\lambda \in \Lambda(T,E)$ such that $(f\delta\lambda)(s,t) \in \mathbb{C}$ for all $s,t \in T$.

b) If $K_1(\mathscr{C}(\Omega, \mathbb{C})) = \{0\}$ for the classical K_1 then $K_1(E) = \{0\}$ for the present theory.

a) follows from Lemma 9.2.12 a).

b) follows from a), Remark 1 of Example 5.0.4, and Proposition 7.1.10. ■

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Subject Index

Symbol Index

δ_1 (Corollary 7.2.3)

CF, SF, θ_F, i_F, j_F, C_φ, S_φ (Definition 7.3.1)

$\widetilde{P}$ (Definition 8.1.2)

ν_F (Proposition 8.1.3)

β_F (Proposition 8.1.4, Proposition 8.1.5)

$Trig(n)$, $Pol(n,m)$, $Pol(n)$, $Lin(n)$, $Proj(n)$ (Definition 8.2.1)

$|p|$, $(p,q)_i$ (Definition 8.2.3)

δ_0 (Corollary 8.3.8 b))

$\Phi(F)$, $\Phi_{i,F}$, $K^{f'}$ (Introduction to section 9.1)

$\mathscr{F}(S,\mathbb{C})$ (Definition 9.2.2)

$\Lambda(S,E)$, $\delta\lambda$ (Definition 9.2.3)

$\mathscr{C}(E;\Gamma,F)$, $\mathscr{C}_0(E;\Omega',F)$ (Definition 9.2.5)

Author's Short Biography

Corneliu Constantinescu is emeritus professor of the Swiss Federal Institute of Technology Zürich. He worked in the Theory of Riemann surfaces, Axiomatic Potential Theory, Spaces of Measures, and C*-algebras and he published books in all these fields.

Short Introduction to the Book

The book consists of two parts. Part I is an axiomatic frame for the K-theory for C*-algebras. Some central results of this theory are heaved to the status of axioms and the other results are then derived from these axioms. In Part II the author constructs an example for this axiomatic theory which generalizes the classical theory for C*-algebras.

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